Solutions | PS#2 | 225b | 1/11/2

$$\tilde{\rho}(\vec{q}) = \frac{1}{\left(1 + \frac{|\vec{q}|^2}{0.71\,(\text{GeV})^2}\right)^2} \qquad b = \frac{1}{0.71}\frac{1}{(\text{GeV})^2}$$

$$= \frac{1}{(1 + b|\vec{q}|^2)^2}$$

since $\rho(\vec{q}) = \int d^3x\, e^{-i\vec{q}\cdot\vec{x}} \rho(\vec{x})$

$$\rightarrow \rho(0) = 1$$

$$\rho(\vec{x}) = \frac{1}{(2\pi)^3} \int d^3q\, e^{i\vec{q}\cdot\vec{x}} \hat{\rho}(\vec{q})$$

$$= \frac{1}{(2\pi)^3} \int_0^\infty d|\vec{q}|\,|\vec{q}|^2 \frac{1}{(1+b|\vec{q}|^2)^2} \int d\phi \int_{-1}^{1} d\mu\, e^{i|\vec{q}|r\mu}$$

$$= \frac{1}{(2\pi)^2} \int_0^\infty \frac{d|\vec{q}|\,|\vec{q}|^2}{(1+b|\vec{q}|^2)^2} \frac{e^{i|\vec{q}|r} - e^{-i|\vec{q}|r}}{i|\vec{q}|r}$$

$$= \frac{1}{2\pi^2 r} \int_0^\infty \frac{d|\vec{q}|\,|\vec{q}| \sin(|\vec{q}|r)}{(1 + b|\vec{q}|^2)^2}$$

$$\int^\infty \frac{x \sin(mx)}{(a^2+x^2)^2} dx = \frac{\pi m}{4a} e^{-ma}$$

$$a = \frac{1}{\sqrt{b}}$$

$$\rho(\vec{x}) = \frac{1}{2\pi^2 \cancel{r}} \frac{1}{b^2} \frac{\pi \cancel{r}}{4/\sqrt{b}} e^{-r/\sqrt{b}}$$

$$\boxed{\rho(\vec{x}) = \frac{1}{8\pi} \frac{1}{b^{3/2}} e^{-r/\sqrt{b}}}$$

$$b = \frac{1}{0.71} \frac{1}{(GeV)^2} = \frac{(0.1973)^2}{0.71} \text{ fm}^2$$

$$= 0.055 \text{ fm}^2$$

$$\sqrt{b} = 0.23 \text{ fm}$$

Check Norm

$$\int_0^\infty d^3x\, \rho(\vec{x}) = \frac{1}{8\pi} \frac{1}{b^{3/2}} \int dr\, r^2 \int_0^{2\pi} d\phi \int_{-1}^{1} d\nu\, e^{-r/\sqrt{b}}$$

$$= \frac{1}{2} \int d\xi\, \xi^2 e^{-\xi} = \frac{2!}{2} = 1$$

RMS

$$\int_0^\infty d^3x\, r^2 \rho(\vec{x}) = \frac{1}{2} b \cdot \int d\xi\, \xi^4 e^{-\xi} = 12\, b$$

$$= 0.66 \text{ fm}^2 \quad \langle r^2 \rangle^{1/2} = \sqrt{0.66} = 0.81 \text{ fm}$$

#2 (a) $s = (k+p)^2 = k^2 + p^2 + 2pk$

$$k^2 \to m_e^2 \simeq 0 \qquad p^2 \to m_p^2 \qquad 2pk \to 2(m_p E - \vec{k}\cdot\vec{0})$$

$$\boxed{s = m_p^2 + 2 m_p E}$$

$$\boxed{\lim_{m_p \to 0} s = 2 m_p E \quad \text{(lowest order)}}$$

(b) $E_{total} = E + m_p$

$$\boxed{\gamma = \frac{E_{total}}{\sqrt{s}} = \frac{E + m_p}{\sqrt{m_p(m_p + 2E)}}}$$

$$\boxed{\lim_{m_p \to 0} \gamma = \frac{E}{\sqrt{2 m_p E}} = \sqrt{\frac{E}{2 m_p}}}$$

$$\beta^2 = 1 - \frac{1}{\gamma^2} = 1 - \frac{m_p^2 + 2 m_p E}{(E + m_p)^2}$$

$$= \frac{E^2 + \cancel{2 m_p E} + \cancel{m_p^2} - \cancel{m_p^2} - \cancel{2 m_p E}}{(E + m_p)^2}$$

$$\boxed{\beta = \frac{E}{E + m_p}} \qquad \boxed{\lim_{m_p \to 0} \beta = 1 - \frac{m_p}{E}}$$

(c) $(k+p) = (\sqrt{s}, \vec{0})$ in c.m. frame

$k(k+p) = \sqrt{s}\, E^*$ (cm frame) $\qquad p(k+p) = \sqrt{s}\, E_p^*$

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Lab Frame:

$k(k+p) = \cancel{k^2} + kp$ $\qquad$ $p(k+p) = pk + p^2$

$m_e \simeq 0$ $\qquad\qquad$ $= m_p E + m_p^2$

$= m_p E = \sqrt{s}\, E^*$ $\qquad\qquad$ $= \sqrt{s}\, E_p^*$

$$\boxed{E^* = \frac{m_p E}{\sqrt{m_p(2E+m_p)}} \qquad E_p^* = \frac{m_p(E+m_p)}{\sqrt{m_p(2E+m_p)}}}$$

note $E^* + E_p^* = \dfrac{m_p(2E+m_p)}{\sqrt{m_p(2E+m_p)}}$

$= \sqrt{m_p(2E+m_p)} = \sqrt{s}$

(makes sense)

$$\boxed{\lim_{m_p \to 0} E^* = \frac{1}{\sqrt{2}}\sqrt{m_p E} = \frac{1}{2}\sqrt{s} \qquad E_p = \frac{1}{\sqrt{2}}\sqrt{m_p E} = \frac{1}{2}\sqrt{s}}$$

(d) $\boxed{|\vec{p}_e^{\,*}| = \sqrt{E^{*2} - m_e^2} = E^* = \dfrac{m_p E}{\sqrt{m_p(2E+m_p)}}}$

$\lim_{m_p \to 0} |\vec{p}_e^{\,*}| = \frac{1}{\sqrt{2}}\sqrt{m_p E} = \frac{1}{2}\sqrt{s}$

$|\vec{p}_p^{\,*}|^2 = E_p^{*2} - m_p^2 = \dfrac{m_p^2(E+m_p)^2 - m_p^2\, m_p(2E+m_p)}{m_p(2E+m_p)}$

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$$= \frac{m_p\left[E^2 + 2\cancel{m_p E} + \cancel{m_p^2} - 2\cancel{m_p E} - \cancel{m_p^2}\right]}{(2E + m_p)}$$

$$\boxed{|\vec{p}_p^{\,*}| = \frac{m_p E}{\sqrt{m_p(2E + m_p)}} = |\vec{p}_{e^+}|}$$

$$\lim_{m_p \to 0} = \frac{1}{\sqrt{2}}\sqrt{m_p E} = \frac{1}{2}\sqrt{s}$$

<u>General Case</u>

particle a of mass $\sqrt{s}$ formed by "fusion" of particles with mass m_b & m_c

$$(k+p)^2 = s = m_b^2 + m_c^2 + 2kp$$

$$kp = \frac{1}{2}(s - m_b^2 - m_c^2)$$

$$k(k+p) = \sqrt{s}\, E_b^* = k^2 + kp$$

$$= \sqrt{s}\, E_b^* = m_b^2 + \frac{1}{2}(s - m_b^2 - m_c^2)$$

$$E_b^* = \frac{s + m_b^2 - m_c^2}{2\sqrt{s}}$$

$$|\vec{p}^{\,*}|^2 = E_b^{*2} - m_b^2 = \frac{s^2 + m_b^4 + m_c^4 + 2sm_b^2 - 2sm_c^2 - 2m_b^2 m_c^2}{4s}$$

$$-\frac{4sm_b^2}{4s}$$

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$$|\vec{p}^*| = \frac{s^2+m_b^4+m_c^4-2sm_b^2-2sm_c^2-2m_b^2m_c^2}{4s}$$

$$= \frac{\lambda(s, m_b^2, m_c^2)}{4s}$$

$$\boxed{\lambda(a,b,c) = a^2+b^2+c^2-2ab-2ac-2bc}$$

check:

$$\lambda(s, m_p^2, 0) = s^2+m_p^4 - 2sm_p^2$$

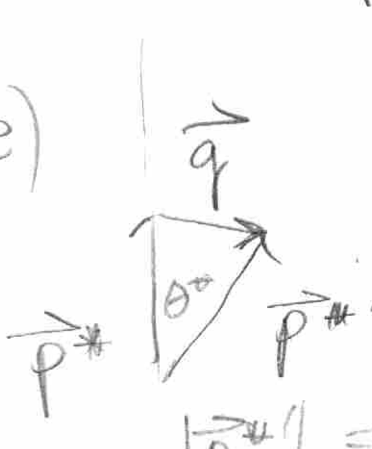
$$= (s-m_p^2)^2$$

$$= (m_p^2+2m_pE - m_p^2)^2 = 4m_p^2E^2$$

$$|\vec{p}^*|^2 = \frac{4m_p^2E^2}{4\,m_p\,(2E+m_p)} = \frac{m_p^2E^2}{m_p(2E+m_p)}$$

$$|\vec{p}^*| = \frac{m_pE}{\sqrt{m_p(2E+m_p)}} \quad \checkmark$$

(e)



$$|\vec{p}^{*\prime}| = |\vec{p}^*|$$

$$|\vec{q}| = 2|\vec{p}^*| \sin\frac{\theta^*}{2}$$

$$\boxed{|\vec{q}| = \frac{2m_pE}{\sqrt{m_p(2E+m_p)}} \sin\frac{\theta^*}{2} \Rightarrow \sqrt{2m_pE}\,\sin\frac{\theta^*}{2} \quad m_p \to 0}$$

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(f) $E' = \gamma(E^* + \beta|\vec{p}|^* \cos\theta^*)$

$$= \frac{E+m_p}{\sqrt{m_p(m_p+2E)}}\left(\frac{m_p E}{\sqrt{m_p(2E+m_p)}} + \frac{E}{E+m_p}\frac{m_p E}{\sqrt{m_p(2E+m_p)}}\cos\theta^*\right)$$

$$= \frac{\cancel{(E+m_p)} m_p E}{\cancel{m_p}(m_p+2E)}\left(\frac{E+m_p+E\cos\theta^*}{\cancel{E+m_p}}\right)$$

$$E' = \frac{E}{m_p+2E}\left(E(1+\cos\theta^*) + m_p\right)$$

$$\nu = E - E' = \frac{-E^2(1+\cos\theta^*) \cancel{- m_p E} + m_p E + 2E^2}{m_p+2E}$$

$$\boxed{\nu = E - E' = \frac{E^2(1-\cos\theta^*)}{m_p+2E} = \frac{2E^2\sin^2\theta^*/2}{m_p+2E}}$$

(g)

$$2m_p\nu = \frac{4m_p E^2 \sin^2\theta^*/2}{m_p+2E}$$

$$\boxed{\lim_{m_p\to 0} \frac{1}{\nu} = E\left(1+\frac{m_p}{2E}\right)\sin^2\theta^*/2}$$

$$-q^2 = +|\vec{q}|^2 = \frac{4m_p^2 E^2}{m_p(2E+m_p)}\sin^2\theta^*/2$$

$$-q^2 = \frac{4m_p E^2 \sin^2\theta^*/2}{m_p+2E}$$

So, yes, $2m_p\nu = -q^2$ ✓

(h) $p_{\parallel} = \gamma(|\vec{p}^*|\cos\theta^* + \beta E^*) = \gamma E^*(\cos\theta^* + \beta)$

$p_{\perp} = |\vec{p}^*|\sin\theta^* = E^*\sin\theta^*$

$$\boxed{\tan\theta = \frac{\cancel{E^*}\sin\theta^*}{\gamma\cancel{E^*}(\beta+\cos\theta^*)} = \frac{\sin\theta^*}{\gamma(\cos\theta^*+\beta)}}$$

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(i) recoil proton ...

$$p_{\parallel} = \gamma(+\cos\theta^* \cancel{\pm}\beta)E^*$$
$$p_{\perp} = |\vec{p}^*|\sin\theta^* = E^*\sin\theta^*$$

$$\tan\phi = \frac{\sin\theta^*}{\gamma(+\cos\theta^* \cancel{\pm}\beta)}$$

Look at $m_p \to 0$ for (h) + (i)

$$\tan\theta \to \frac{\sin\theta^*}{\gamma(\cos\theta^* + \frac{E}{E+m_p})}$$

$$\to \frac{\sin\theta^*}{\gamma(\cos\theta^* + 1 - \frac{m_p}{E})}$$

$$\to \frac{\sin\theta^*}{\gamma(2\cos^2\frac{\theta^*}{2})(1 - \frac{m_p}{2\cos^2\frac{\theta^*}{2}})}$$

$$\to \frac{2\sin\theta^*/2\cos\theta^*/2}{\gamma\cdot 2\cos^2\theta^*/2}\left(1 + \frac{m_p}{2\cos^2\theta^*/2}\right)$$

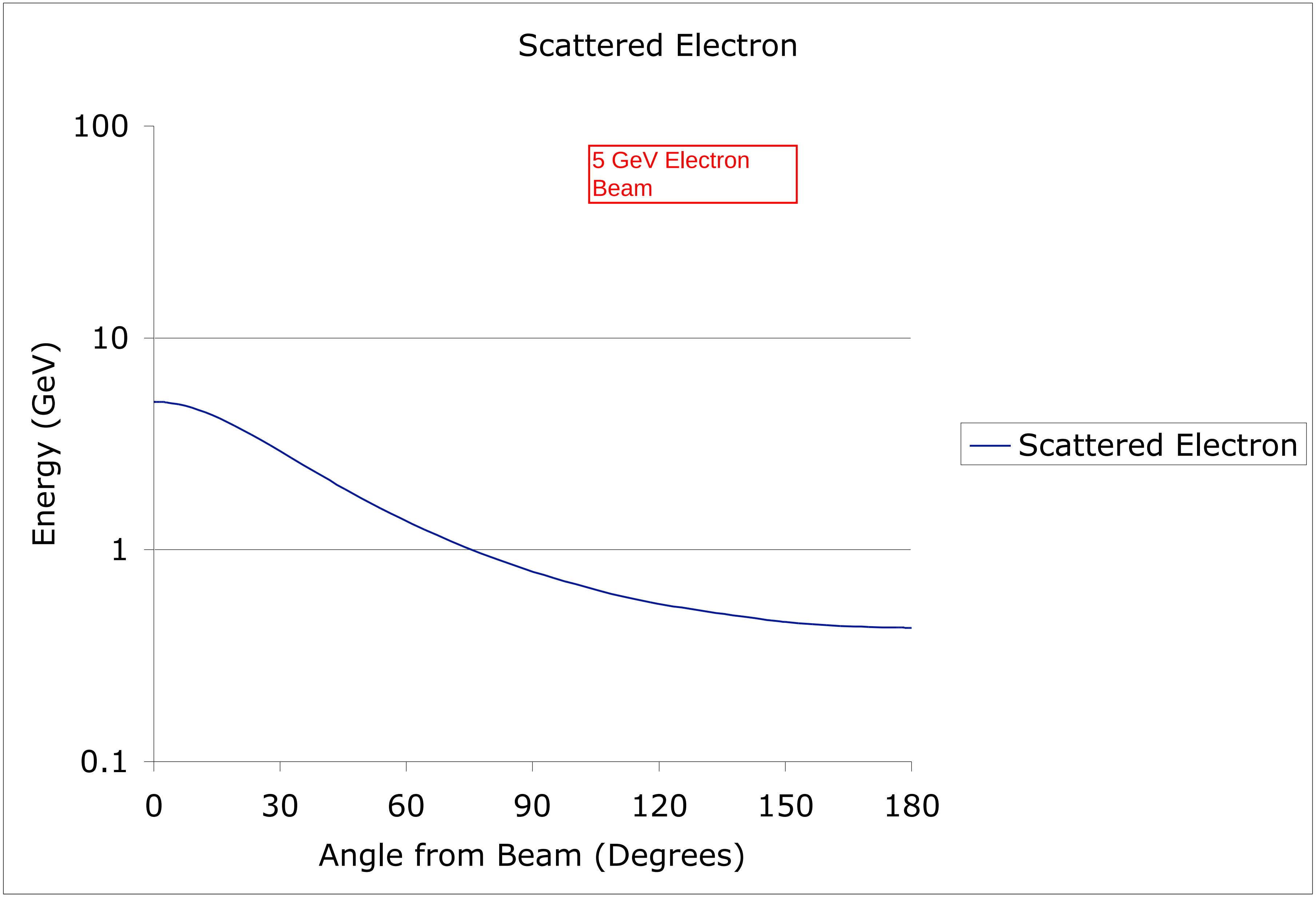
$$\boxed{\tan\theta \to \frac{1}{\gamma}\tan\theta^*/2\cdot\left(1 + \frac{m_p}{2\cos^2\theta^*/2}\right)}$$

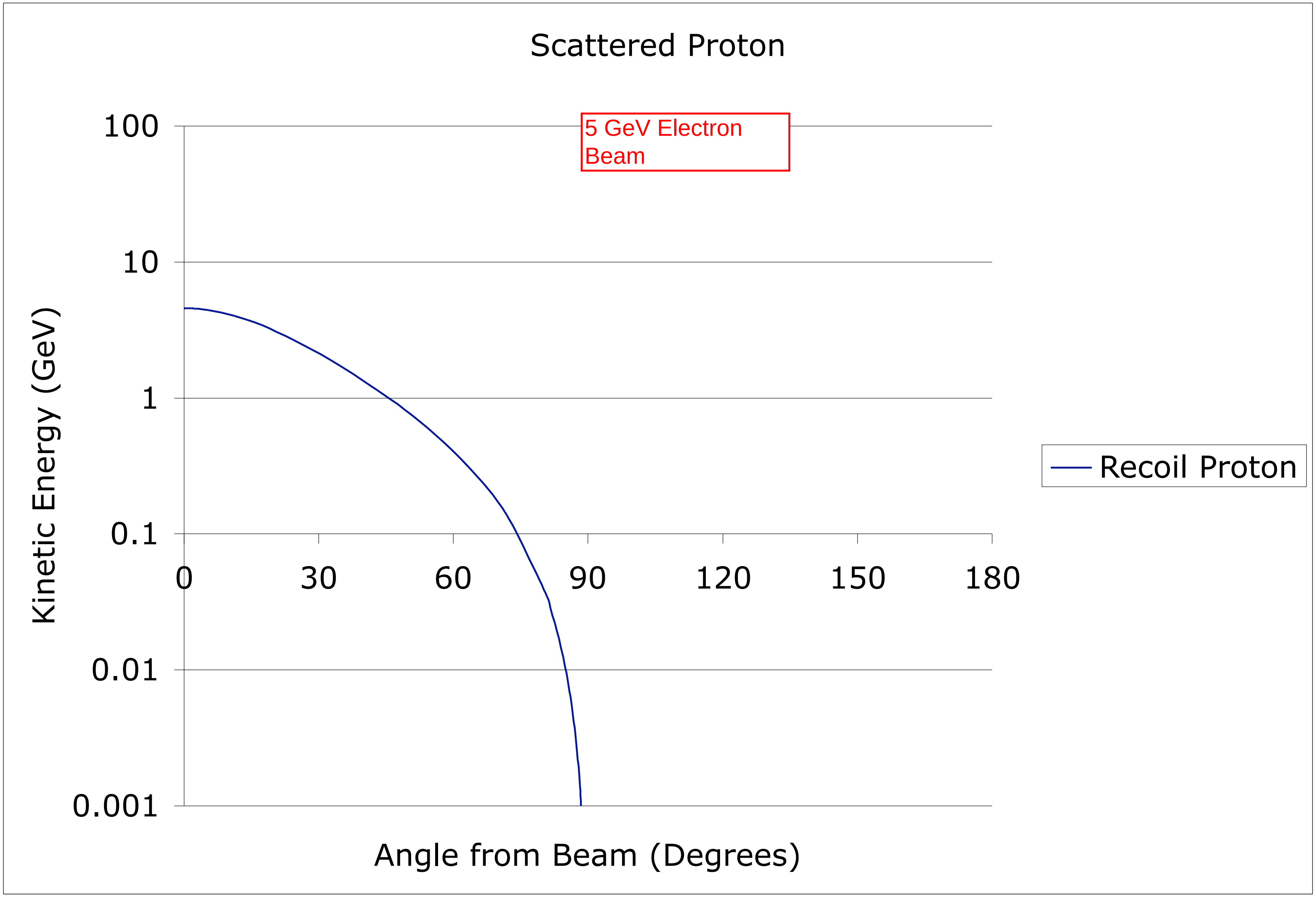
$$\tan\phi \to \frac{\sin\theta^*}{\gamma(\cos\theta^* - \beta)} = \frac{\sin\theta^*}{\gamma(\cos\theta^* - \frac{E}{E+m_p})}$$

$$\Rightarrow \frac{\sin\theta^*}{\gamma(\cos\theta^* - 1 + m_p/E)}$$

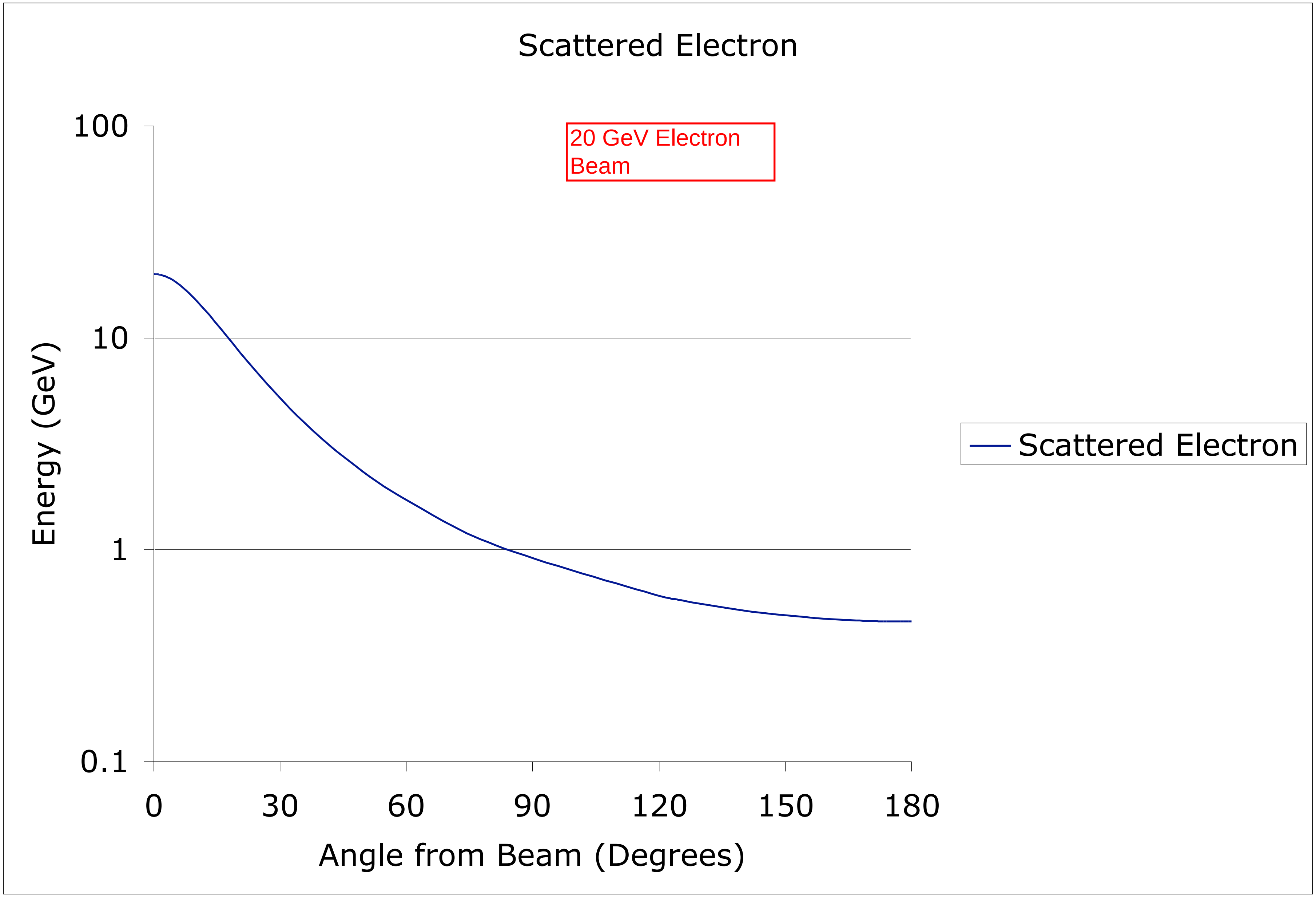
$$\Rightarrow \frac{2\sin\theta^*/2\cos\theta^*/2}{\gamma(-2\sin^2\frac{\theta^*}{2})(1 - m_p/2\sin^2\theta^*/2)}$$

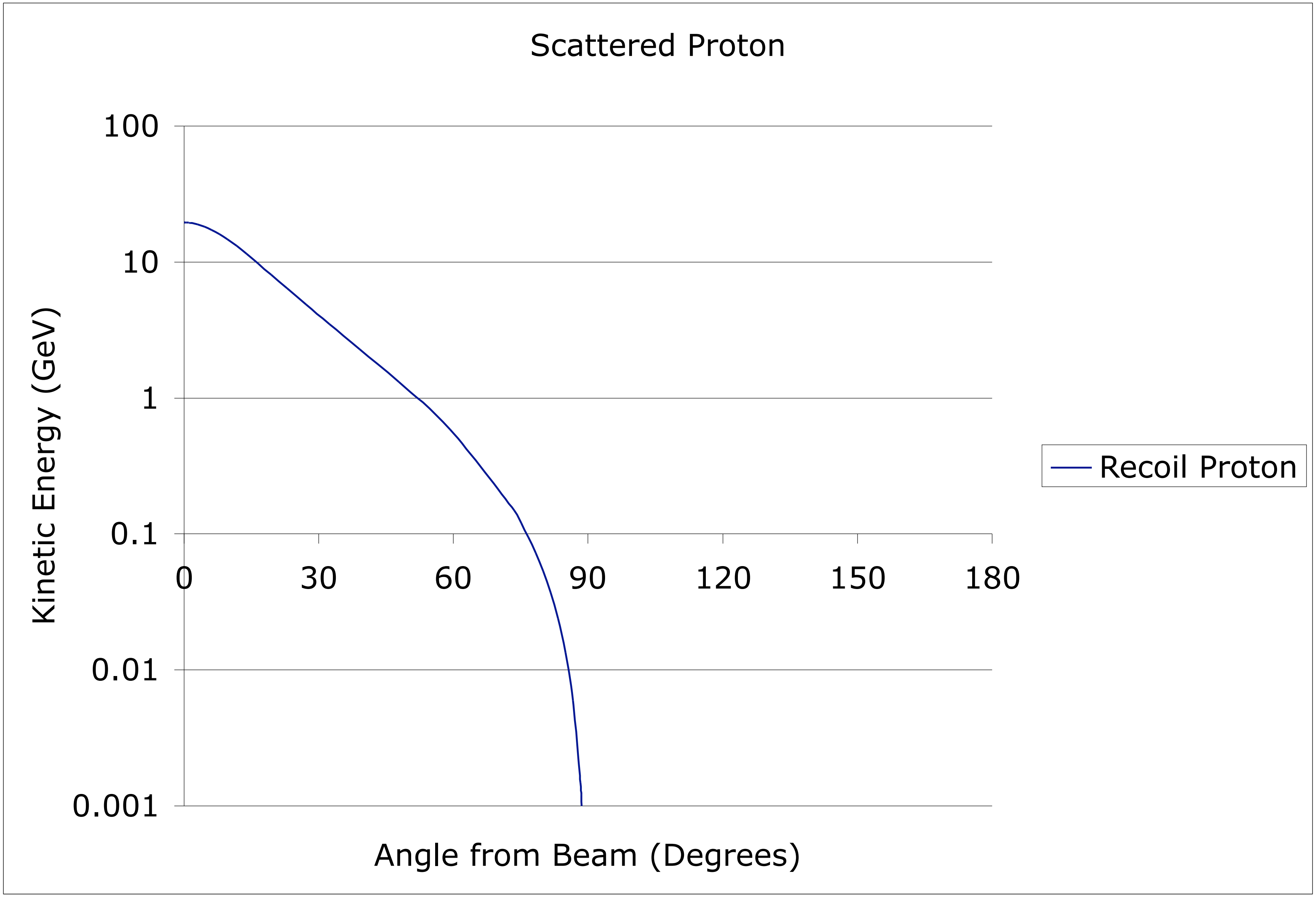
$$\boxed{\tan\phi \Rightarrow -\frac{1}{\gamma}\frac{1}{\tan\theta^*/2}\left(1 + \frac{m_p}{2\sin^2\theta^*/2}\right)}$$


Scattered Electron
100
10
1
0.1
Energy (GeV)
5 GeV Electron
Beam
0
30
60
90
120
150
180
Angle from Beam (Degrees)
Scattered Electron



Scattered Proton
5 GeV Electron Beam
100
10
1
0.1
0.01
0.001
Kinetic Energy (GeV)
0
30
60
90
120
150
180
Angle from Beam (Degrees)
Recoil Proton


Scattered Electron
20 GeV Electron Beam
100
10
1
0.1
Energy (GeV)
0
30
60
90
120
150
180
Angle from Beam (Degrees)
Scattered Electron


Scattered Proton
100
10
1
0.1
0.01
0.001
0
30
60
90
120
150
180
Kinetic Energy (GeV)
Angle from Beam (Degrees)
Recoil Proton