P22 HW 7

10.10

Recall Q is $\frac{\text{total } E \text{ of oscillator}}{E \text{ lost in } 1 \text{ rad}} \doteq \frac{\omega}{\gamma}$

So for our clock powered by the gravitational potential energy of the falling weight, we have $E = Mgh = (.2\,kg)g(2\,m) = 3.92\,J$ which is uniformly dissipated over the course of 1 day to power the clock to maintain $T = 1\,s$

Since we know our clock is lightly damped (or else it would be a crummy clock), we know $Q \approx \frac{\omega_0}{\gamma} = \frac{\sqrt{g/\ell}}{\gamma} = \frac{6.26\ {}^{1}/_{s}}{\gamma}$

We know for the oscillator, it starts w/ amplitude $\theta = .2$ rad $\Rightarrow E_0 = U = mgy = mg(\ell - \ell\cos\theta)$

$$= (.01\,kg)g(.25\,m)[1 - \cos(.2)]$$

$$E_0 = 4.88 \times 10^{-4}\,J$$

So we also know our (weakly damped) oscillator loses E related to $\frac{dE}{dt} = -\gamma E$

So the rate that the weight powers the clock must be equal to the energy dissipated,

$$P = -\frac{\Delta E}{\Delta t} = \frac{-3.92\,J}{24\,hr \times 3600\,{}^{s}/_{hr}} = -4.54 \times 10^{-5}\,W \equiv -\gamma E_0$$

$$\gamma = \frac{P}{E_0} = \frac{4.54 \times 10^{-5}\,W}{4.88 \times 10^{-4}\,J} = 0.093\,s^{-1} \Rightarrow \boxed{Q \approx \frac{\sqrt{g/\ell}}{\gamma} = 67.3}$$

or for a battery providing $1\,J$, since we need $3.92\,J$ for 1 day: $\boxed{\frac{1\,J}{3.92\,{}^{J}/_{day}} = 0.266\ \text{day} = 6.12\ \text{hr of functionality}}$

(10.3) Undamped: $x(t) = A\sin\omega_0 t$, $v(t) = A\omega_0\cos\omega_0 t$

see max displacement $(v=0)$ at $\omega_0 t = \pi/2$

For damped, we have $x(t) = Ae^{-\gamma t/2}\cos(\omega t + \phi)$ and if we are lightly damped, i.e. $Q \gg 1$ as is stated in the problem, then we can approximate $\omega \approx \omega_0$

Now finding the maximum; $v = 0$

$$v(t) = Ae^{-\gamma t/2}\left[-\frac{\gamma}{2}\cos(\omega t + \phi) - \omega\sin(\omega t + \phi)\right] = 0$$

$$-\frac{1}{2}\frac{\gamma}{\omega}\cos(\omega t + \phi) = \sin(\omega t + \phi)$$

$$\text{or} \quad -\frac{1}{2}\frac{1}{Q} = \tan(\omega t + \phi)$$

Now since Q is large, the left hand side is small, which corresponds to a tangent argument that is small $(\tan(0) = 0)$, or some integer multiple of π which isn't any more informative, so expanding the small argument x

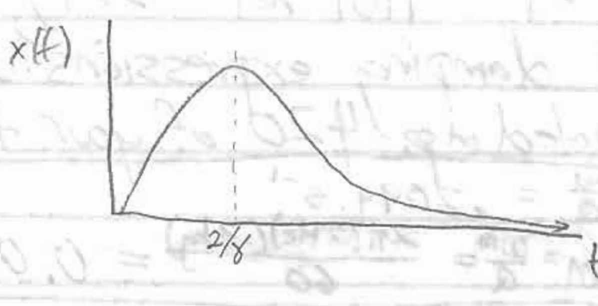
$\tan x \approx x$ (from Taylor expansions)

$$\cancel{\omega t} + \phi = -\frac{1}{2Q} \quad \Rightarrow \quad \phi = -\frac{1}{2Q}$$

The minus sign signifies that there is a shift to the left in time, indicative that the maximum occurs <u>earlier</u> than in the undamped case.

(10.3) We know max displacement corresponds to $v=0$, so

$$v(t) = 0 = \frac{I}{m} e^{-\gamma t/2}\left[-\frac{\gamma}{2}t + 1\right] \Rightarrow \boxed{t = \frac{2}{\gamma}}$$



Problem 5 We know $c = 299792458 \frac{m}{s}$ or
$c = 670616629.4 \frac{\text{miles}}{\text{hr}}$

a) $t = \frac{\Delta x}{c} = \frac{85 \text{ miles}}{c} = 1.267 \times 10^{-7} \text{ hr} \times \frac{3600 s}{1 \text{ hr}} = 4.56 \times 10^{-4} s$

b) $t = \frac{\Delta x}{c} = \frac{277 \text{ miles}}{c} = 4.131 \times 10^{-7} \text{ hr} = 1.49 \times 10^{-3} s$

c) $t = \frac{\Delta x}{c} = \frac{1635 \text{ miles}}{c} = 2.438 \times 10^{-6} \text{ hr} = 8.78 \times 10^{-3} s$

d) $t = \frac{\Delta x}{c} = \frac{8666 \text{ miles}}{c} = 1.292 \times 10^{-5} \text{ hr} = 4.65 \times 10^{-2} s$

e) $t = \frac{10443 \text{ miles}}{c} = 1.557 \times 10^{-5} \text{ hr} = 5.61 \times 10^{-2} s$

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(10.2) Q is defined as $\frac{E \text{ stored in oscillator}}{E \text{ dissipated per radian}}$

where the time to oscillate through 1 rad is $\frac{T}{2\pi} = \frac{1}{\omega}$

So given a $Q = 60 > 1$ (but not $>> 1$), we can use the light damping expressions w/ minimal error (as noted on p. 420 of your texts)

$$\text{So } Q \approx \frac{\omega}{\gamma} = \frac{\omega}{(b/m)} \Rightarrow \boxed{\begin{array}{l}\gamma = \frac{\omega}{Q} = .2094 \text{ s}^{-1} \\ b = \gamma m = \frac{\omega m}{Q} = \frac{2\pi(2\text{ Hz})(.3\text{ kg})}{60} = 0.063 \frac{\text{kg}}{\text{s}}\end{array}}$$

$$\omega^2 = \omega_0^2 - \left(\frac{\gamma}{2}\right)^2 = (4\pi)^2 \Rightarrow \frac{k}{m} = (4\pi)^2 + \left(\frac{\gamma}{2}\right)^2$$

$$\boxed{k = m\left(16\pi^2 + \frac{\gamma^2}{4}\right) = 47.4 \ \frac{N}{m}}$$

(10.5) $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$ + have $x = (A + Bt)e^{-(\gamma/2)t}$

$$\dot{x} = e^{-\gamma/2 t}\left[\frac{-\gamma}{2}(A + Bt) + B\right] = -\frac{\gamma}{2}x + Be^{-\gamma/2 t}$$

$$\ddot{x} = -\frac{\gamma}{2}\dot{x} - \frac{\gamma}{2}Be^{-\gamma/2 t} = \left(-\frac{\gamma}{2}\right)\left[\frac{-\gamma}{2}x + Be^{-\gamma/2 t} + Be^{-\gamma/2 t}\right]$$

Adding: $\left(\frac{\gamma}{2}\right)^2 x - 2\left(\frac{\gamma}{2}\right)Be^{-\gamma/2 t} + \gamma\left[-\frac{\gamma}{2}x + Be^{-\gamma/2 t}\right] + \omega_0^2 x = 0$

$$\underbrace{\left(\left(\frac{\gamma}{2}\right)^2 - \frac{\gamma^2}{2} + \omega_0^2\right)} x + \cancel{(\gamma B - \gamma B)e^{-\gamma t/2}} = 0$$

But $\gamma = 2\omega_0 \Rightarrow \frac{-\gamma^2}{4} + \omega_0^2 = -\omega_0^2 + \omega_0^2 = 0$!

so it checks.. Now solving for A + B:

$x(0) = 0 \Rightarrow A = 0$, + for impulse $I \Rightarrow B = \frac{I}{m}$

so $x(t) = Bte^{-\gamma t/2} = \frac{I}{m}te^{-\gamma t/2}$, $v(t) = \frac{I}{m}\left[-\frac{\gamma}{2}t + 1\right]e^{-\gamma t/2}$