HW 6 PHYS 22

① We have a damped spring, so we modify Newton's eqn. accordingly (damping proportional to speed $\dot{x}$)

$$F = ma = m\ddot{x} = -kx - b\dot{x}$$

To solve this differential equation:

$$m\ddot{x} + b\dot{x} + kx = 0 \implies \ddot{x} + \left(\frac{b}{m}\right)\dot{x} + \left(\frac{k}{m}\right)x = 0$$

Or let $\tilde{b} \equiv \left(\frac{b}{m}\right)\left(\frac{1}{2}\right)$ + $\omega_0^2 \equiv \frac{k}{m}$

$$\implies \ddot{x} + (2\tilde{b})\dot{x} + \omega_0^2 x = 0$$

Now guess a solution of the form $x(t) = x_0 e^{rt}$

Plugging this in, we get the following equation, usually referred to as the characteristic eqn:

$$(r^2 + (2\tilde{b})r + \omega_0^2)x_0 e^{rt} = 0$$

Solving: $r = -\frac{2\tilde{b}}{2} \pm \sqrt{\left(\frac{2\tilde{b}}{2}\right)^2 - \frac{4}{4}\omega_0^2} = -\tilde{b} \pm \sqrt{\tilde{b}^2 - \omega_0^2}$

So our solution before applying the initial conditions of $x(0) = 0$ and $\dot{x}(0) = .2\ \frac{m}{s}$ is

$$x(t) = Ae^{(-\tilde{b}+\sqrt{\tilde{b}^2-\omega_0^2})t} + Be^{(-b-\sqrt{\tilde{b}^2-\omega_0^2})t} = e^{-\tilde{b}t}\left[Ae^{t\sqrt{\tilde{b}^2-\omega_0^2}} + Be^{-t\sqrt{\tilde{b}^2-\omega_0^2}}\right]$$

$$x(0) = 0 = A + B \implies \boxed{A = -B}$$

$$\dot{x}(t) = \left(-\tilde{b} + \sqrt{\tilde{b}^2-\omega_0^2}\right)Ae^{(-\tilde{b}+\sqrt{\tilde{b}^2-\omega_0^2})t} + \left(-\tilde{b} - \sqrt{\tilde{b}^2-\omega_0^2}\right)(-A)e^{(-\tilde{b}-\sqrt{\tilde{b}^2-\omega_0^2})t}$$

$$\dot{x}(0) = \left(-\tilde{b} + \sqrt{\tilde{b}^2-\omega_0^2} + \tilde{b} + \sqrt{\tilde{b}^2-\omega_0^2}\right)A = 2\left(\sqrt{\tilde{b}^2-\omega_0^2}\right)A = .2$$

So from $\dot{x}(0) = .2 = 2\left(\sqrt{\tilde{b}^2 - \omega_0^2}\right) A$ we find

$$\boxed{A = .1\left(\tilde{b}^2 - \omega_0^2\right)^{-1/2}}$$

$$\boxed{x(t) = \frac{.1}{\sqrt{(\tilde{b}^2 - \omega_0^2)}}\, e^{-\tilde{b}t}\left[e^{t\sqrt{\tilde{b}^2-\omega_0^2}} - e^{-t\sqrt{\tilde{b}^2-\omega_0^2}}\right]}$$

Now we put in the numbers + examine the cases, where $\omega_0^2 = \frac{k}{m} = \frac{0.4\ N/m}{0.1\ kg} = 4\ \frac{1}{s^2}$

a) $b = 5\times10^{-3}\ \frac{kg}{s} \Rightarrow \tilde{b} = \frac{1}{2}\frac{b}{m} = \frac{1}{2}\left(.05\ \frac{1}{s}\right) = .025\ \frac{1}{s}$

so our time constant $\tau = \frac{m}{b} = \frac{1}{.05} = 20\ s$

and $\tilde{b}^2 < \omega_0^2 \Rightarrow$ we are underdamped

(Ex) $x(t) = (-.05i)e^{-.025t}\left[e^{1.99it} - e^{-i(1.99)t}\right] = (.1)e^{-(.025)t}\sin(1.99t)$

b) $b = 0.4\ \frac{kg}{s} \Rightarrow \tilde{b} = \frac{b}{2m} = \frac{1}{2}\left(4\ \frac{1}{s}\right) = 2\ \frac{1}{s}$

$\tau = \frac{m}{b} = \frac{1}{4} = 0.25\,s$; $\tilde{b}^2 = \omega_0^2 \Rightarrow$ we are crit. damped

to solve for $x(t) = Ce^{-\tilde{b}t} + Dte^{-\tilde{b}t}$ $\Rightarrow x(0) = 0 \Rightarrow C = 0$

$\dot{x}(0) = 0.2 \Rightarrow D = 0.2$, so $x(t) = 0.2\,te^{-2t}$

c) $b = 0.2\sqrt{20}\ \frac{kg}{s} \Rightarrow \tilde{b} = \frac{b}{2m} = \sqrt{20}\ \frac{1}{s}$

$\tau = \frac{m}{b} = \frac{1}{8.94} = 0.112\ s$; $x(t) = \frac{1}{40}e^{-\sqrt{20}t}\left[e^{4t} - e^{-4t}\right]$

$\tilde{b}^2 > \omega_0^2 \Longrightarrow$ We are overdamped