More Strongly Damped Systems

$$Q = \frac{\omega_0}{\gamma} \lesssim 10 \quad \text{or} \quad 10^{-3}!$$

$$m\ddot{z} + b\dot{z} + kz = 0 \qquad z = z_0 e^{\alpha t}$$

$$\downarrow$$

$$\alpha = -\frac{1}{2}\frac{b}{m} \pm \sqrt{-\frac{k}{m} + \left(\frac{b}{2m}\right)^2}$$

$$\alpha = -\frac{1}{2}\gamma \pm \sqrt{-\omega_0^2 + \frac{1}{4}\gamma^2}$$

$\uparrow$

fun part

imagine γ getting bigger + bigger

crucial boundary $\quad -\omega_0^2 + \frac{1}{4}\gamma^2 = 0!$

$$\gamma = 2\omega_0$$

$$\frac{b}{m} = 2\sqrt{\frac{k}{m}}$$

"critical damping"

$$\alpha = -\frac{1}{2}\gamma t$$

but $\quad x = x_0 e^{-\frac{1}{2}\gamma t}$

only one arbitrary constant

how do you handle $x(0)$, $\dot{x}(0)$?

→ there is another solution, a "remnant"

$$m\ddot{z} + b\dot{z} + kz = 0$$
$$\ddot{z} + \gamma\dot{z} + \omega_0^2 z = 0$$

critical $\gamma = 2\omega_0$, $Q = \frac{\omega_0}{\gamma} = \frac{1}{2}$

$z_1 e^{-\frac{\gamma}{2}t}\cos(\omega t)$ $\qquad$ $z_2 e^{-\frac{\gamma}{2}t}\sin(\omega t)$

vanishing

→ $z_1 e^{-\gamma t/2}$ is limit $\qquad$ $z_2 e^{-\gamma t/2} \times t$ is remnant

→ try $\quad z = z_2 e^{-\gamma t/2}\, t$

$$\dot{z} = z_2 e^{-\gamma t/2} - \frac{\gamma}{2} z_2 e^{-\gamma t/2}\, t$$

$$\ddot{z} = -\frac{\gamma}{2} z_2 e^{-\gamma t/2} - \frac{\gamma}{2} z_2 e^{-\gamma t/2} + t\left(\frac{\gamma}{2}\right)^2 z_2 e^{-\gamma t/2}$$

$$= -\gamma z_2 e^{-\gamma t/2} + t\left(\frac{\gamma}{2}\right)^2 z_2 e^{-\gamma t/2}$$

Plug into $\ddot{z} + \gamma \dot{z} + \omega_0^2 z = 0$

$$z_2 e^{-\gamma t/2}\left[-\gamma + \left(\frac{\gamma}{2}\right)^2 t + \cancel{+\gamma} - \frac{1}{2}\gamma^2 t + \underset{\uparrow \atop \frac{\gamma^2}{4}}{\omega_0^2}\, t\right] =$$

$$z_2 e^{-\gamma t/2}\left[\gamma^2 \underbrace{\left(\frac{1}{4} - \frac{1}{2} + \frac{1}{4}\right)}_{0} t\right]$$

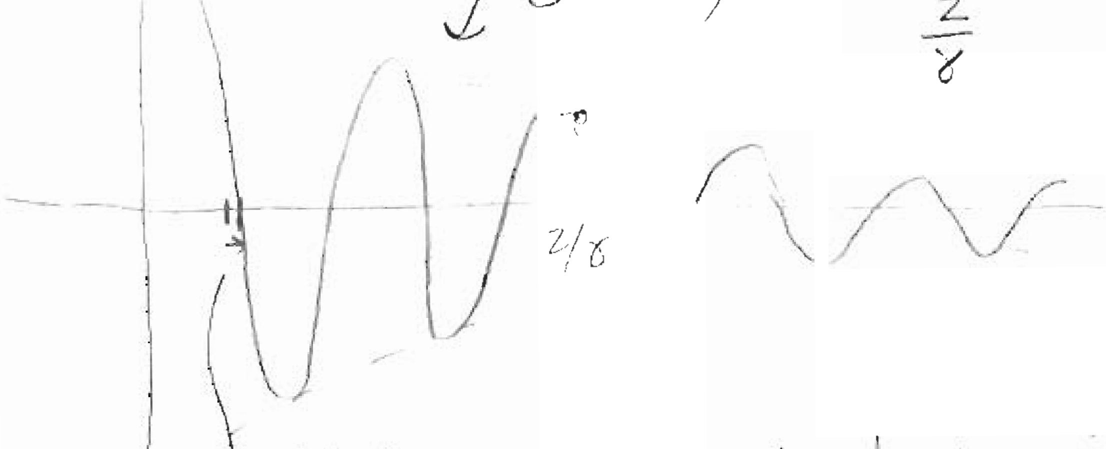
$$= 0 \;!$$

Overdamp

$$\alpha_{\pm} = -\frac{1}{2}\gamma \pm \sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2}$$

γ dominates

$\propto e^{-\frac{1}{2}\gamma t}$, t to fall 1/e is $\frac{2}{\gamma}$



abt longer period due to damping

Energy in Underdamped Oscillator

$$E = \frac{1}{2}kx^2 + \frac{1}{2}m\dot{x}^2$$

$$x = Ae^{-\frac{\gamma}{2}t}\cos(\omega_1 t + \phi) \qquad i\omega_1 = \sqrt{-\omega_0^2 + \tfrac{1}{4}\gamma^2}$$

$$\omega_1 \simeq \left(\omega_0 - \frac{\gamma^2}{8\omega_0}\right)$$

second order

$$\dot{x} = -\frac{\gamma}{2}Ae^{-\frac{\gamma t}{2}}\cos(\omega_1 t + \phi) - \omega_1 A e^{-\gamma t/2}\sin(\omega_1 t + \phi)$$

a little more complicated!

$$\frac{1}{2}kx^2 = \frac{1}{2}kA^2 e^{-\gamma t}\cos^2(\omega_1 t + \phi)$$

$$\frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\omega_1^2 A^2 e^{-\gamma t}\left[\sin^2(\omega_1 t + \phi) + \left(\frac{\gamma}{\omega_1}\right)\sin(\omega_1 t + \phi)\cos(\omega_1 t + \phi) + \left(\frac{\gamma}{2\omega_1}\right)^2\cos^2(\omega_1 t + \phi)\right]$$