Gravity

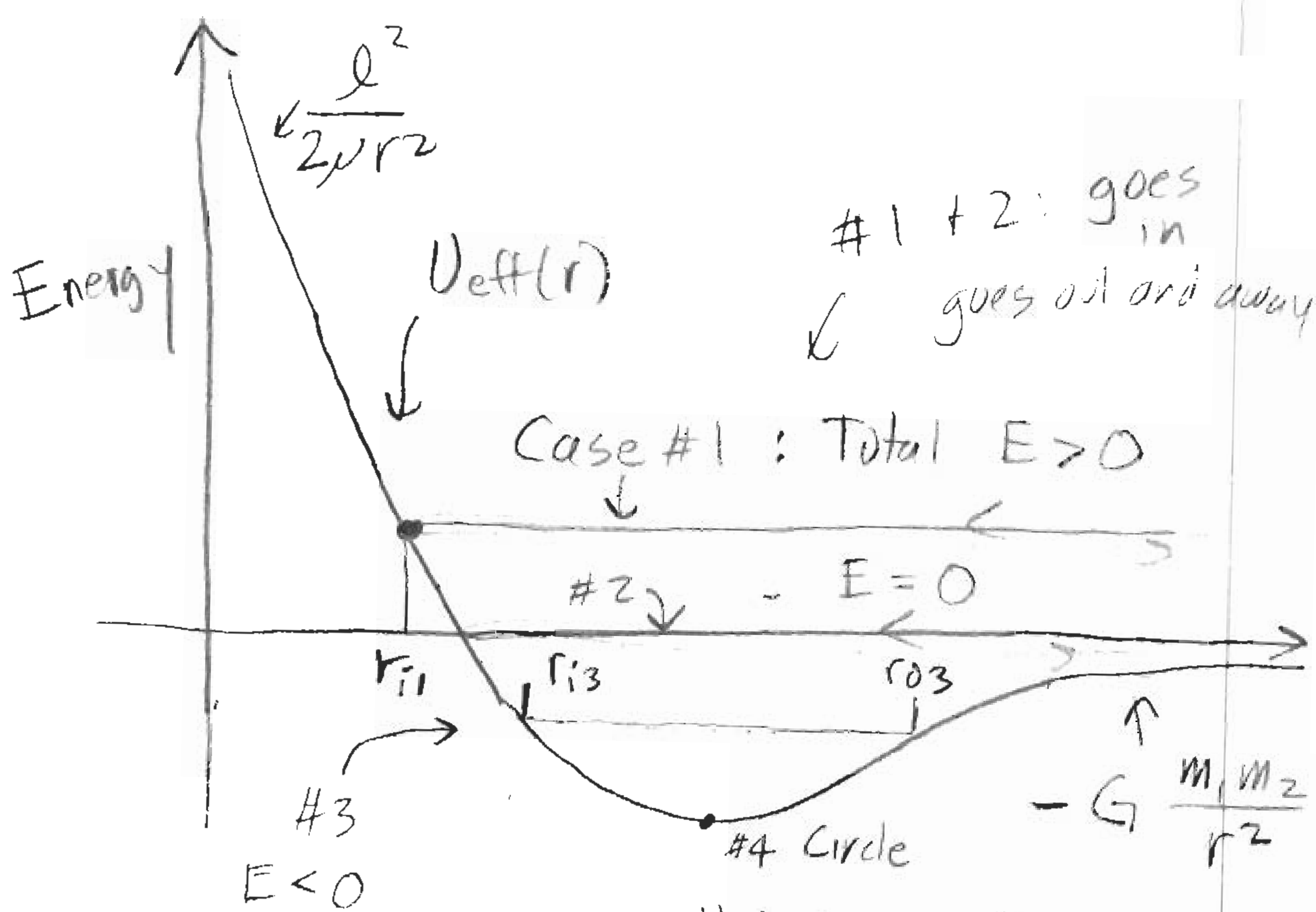
$$f(r) = -G\frac{m_1 m_2}{r^2} \rightarrow U(r) = -G\frac{m_1 m_2}{r}$$

$$U_{eff}(r) = \underbrace{-G\frac{m_1 m_2}{r}}_{\text{attractive}} + \underbrace{\frac{\ell^2}{2\mu r^2}}_{\text{repulsive}}$$

"centrifugal barrier"

$\Rightarrow$ must specify ℓ !!!

$$E = \frac{1}{2}\mu \dot{r}^2 + U_{eff}(r)$$



closed orbit

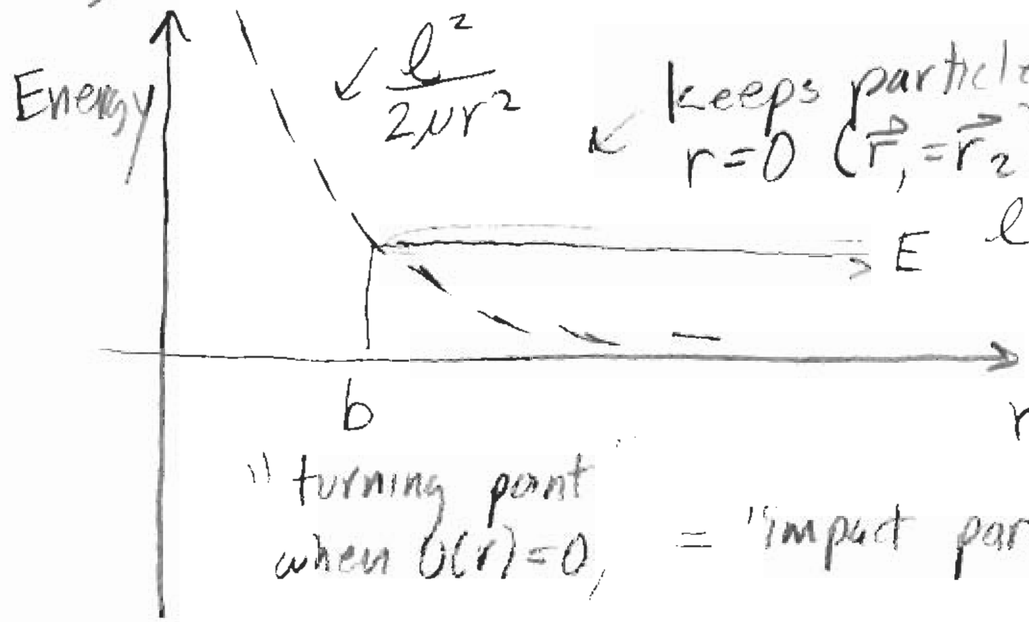
r_{i1}, r_{i2} = "distance of closest approach"

$r_{i3} \rightarrow$ "perihelion perigee", r[illegible] $\rightarrow$ "aphelion apogee"

Mathematica L2482-2694

Qualitative Discussion(s)

$\frac{\ell^2}{2\mu r^2}$ is repulsive, "centrifugal barrier"



keeps particles from $r=0$ ($\vec{r}_1=\vec{r}_2$) unless $\ell=0$

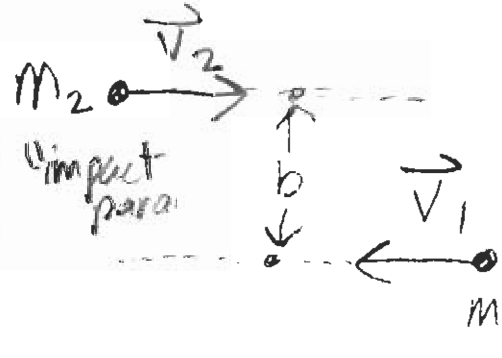
"turning point when $U(r)=0$" = "impact parameter"

$$U_{eff}(r) = U(r) + \frac{\ell^2}{2\mu r^2}$$

Bound States : Orbits (like earth around sun)

$U(r) < 0$ to make this happen.

First $U(r)=0$ think about this



$\vec{L} = ?$ $\quad \vec{r} = \vec{r}_1 - \vec{r}_2$

$\vec{v} = \dot{\vec{r}}_1 - \dot{\vec{r}}_2$

$\vec{v}_0 = \vec{v}_1 - \vec{v}_2$ constant

Center of mass $m_1\vec{v}_1 + m_2\vec{v}_2 = 0$ $\vec{v}_1 \parallel \vec{v}_2$

For <u>fictitious</u> 1-body; want $\perp$ to $\vec{v}$ $\parallel$ to $\vec{v}_1, \vec{v}_2$

$$\vec{L} = \mu \vec{r} \times \vec{v} = \mu(\vec{r}_1 - \vec{r}_2) \times (\vec{v}_1 - \vec{v}_2)$$

$$|\vec{L}| = \ell = \mu b |\vec{v}_1 - \vec{v}_2| = \mu b v_0$$ (direction from

$$\frac{\ell^2}{2\mu r^2} = \frac{\mu^2 b^2 v_0^2}{2\mu r^2} = \frac{1}{2}\mu v_0^2 \times \left(\frac{b}{r}\right)^2$$

$$= (\text{Kinetic Energy}) \cdot \left(\frac{b}{r}\right)^2$$

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\mu v_0^2 \cdot \left(\frac{b}{r}\right)^2$$

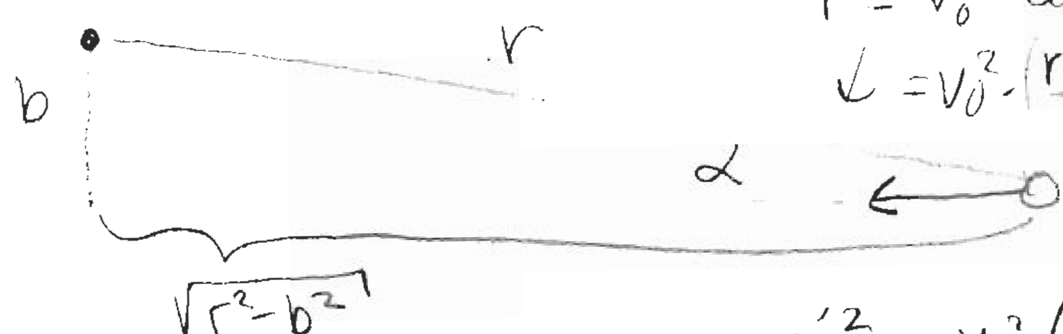
$t = 0 \ldots$ $\quad E = \frac{1}{2}\mu v_0^2 = \frac{1}{2}\mu \dot{r}^2 \rightarrow \dot{r} = \pm v_0$

$\dot{r} = v_0$

$$\frac{1}{2}\mu v_0^2 = \frac{1}{2}\mu \dot{r}^2 \cdot \frac{1}{2}\mu v_0^2 \left(\frac{b}{r}\right)$$

$$\dot{r}^2 = v_0^2\left(1 - \frac{b^2}{r^2}\right)$$

"Fictitious" Diagram

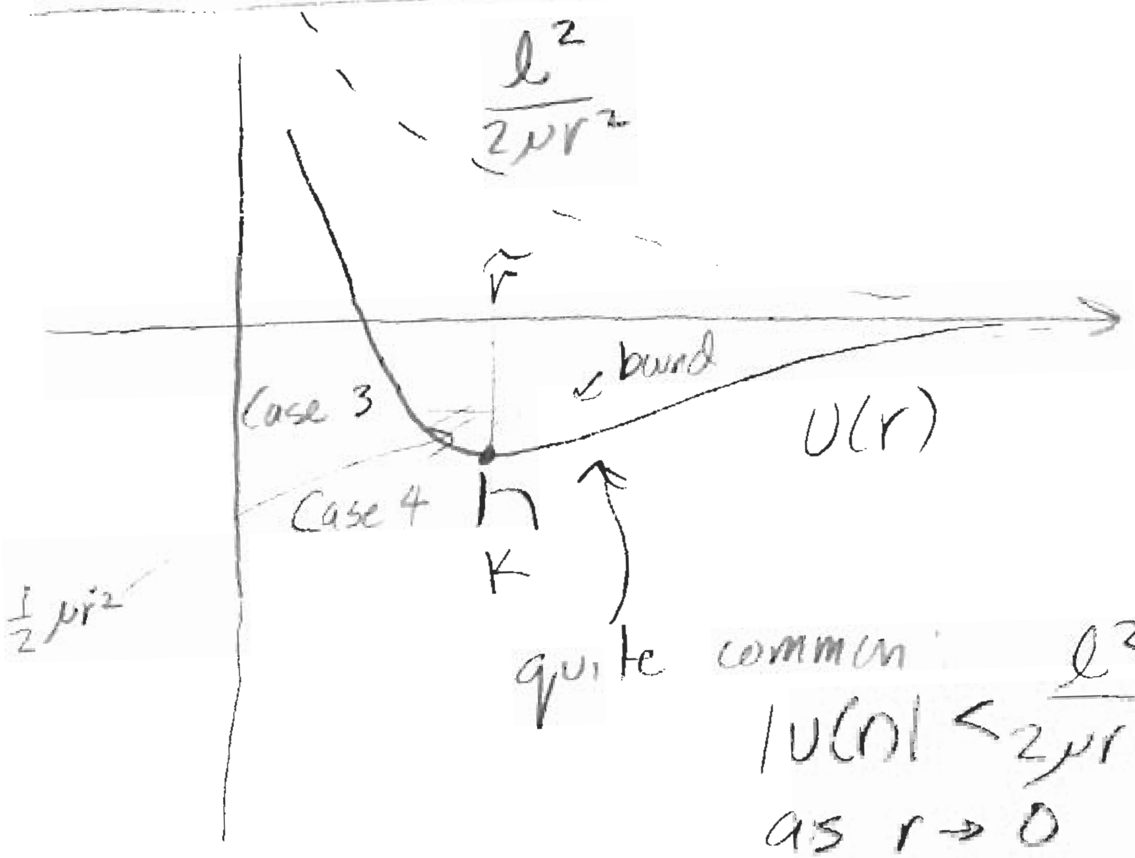


$$\dot{r}^2 = v_0^2 \cdot \cos^2\alpha$$
$$= v_0^2 \cdot \frac{r^2 - b^2}{r^2}$$

$$\dot{r}^2 = v_0^2\left(1 - \frac{b^2}{r^2}\right)$$

$\dot{r} = 0$ when $r = b$ $\quad \dot{r} < 0$ to $\dot{r} > 0$

Introduce Attractive $U(r)$



$|U(r)| < \frac{\ell^2}{2\mu r^2}$

as $r \to 0$

Barrier Wins!

Case 4: Minimum Energy (<0)

Always a circular orbit! (one radius)

$$\left.\frac{dU}{dr}\right|_{\tilde{r}} - \frac{\ell^2}{\mu \tilde{r}^3} = 0$$

$$U(r) = \frac{-A}{r^n} \quad A>0$$

$$\frac{dU}{dr} = \frac{+nA}{r^{n+1}}$$ KK 9.4

$$\frac{nA}{\tilde{r}^{n+1}} - \frac{\ell^2}{\mu \tilde{r}^3} = 0$$

$$\frac{\mu n A}{\ell^2} = \tilde{r}^{n-2}$$

$$\boxed{\tilde{r} = \left(\frac{\mu n A}{\ell^2}\right)^{\frac{1}{n-2}}}$$

Stability: $\frac{d^2 U_{eff}}{dr^2} = \frac{-n(n+1)A}{r^{n+2}} + \frac{3\ell^2}{\mu \tilde{r}^4} > 0$ minimum

<u>Trick</u>:
$$\frac{1}{\tilde r}\frac{nA}{\tilde r^{n+1}} = \frac{\ell^2}{\mu \tilde r^3}\cdot\frac{1}{\tilde r} \equiv \beta > 0$$

$$\frac{d^2 U_{eff}}{dr^2} = (3-n-1)\cdot\beta$$
$$= (2-n)\beta > 0$$
when $n < 2$

$$\beta = \left(\frac{\ell^2}{\mu}\right)^{\frac{n+2}{n-2}}\cdot(nA)^{\frac{-4}{n-2}}$$

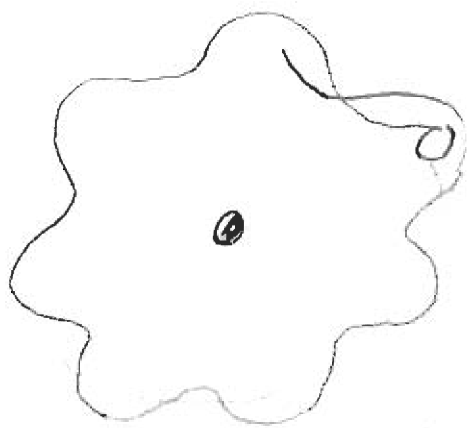
$$U(r) = -\frac{A}{r^n}$$

$n<0$: "stable circular orbit"

$n \geq 2$ "fall to the center" unstable.

<u>Case 3</u>:

Non circular bound state, as oscillation about circular.



does it close?

$$\ell = \mu r^2 \omega_{orbit}$$

$$\omega_{orbit} = \frac{\ell}{\mu r^2}$$

$$\omega_{oscillation} = \sqrt{\frac{1}{\mu}\cdot\left|\frac{d^2 U_{eff}}{dr^2}\right|}$$

will close when

$$\frac{\omega_{oscillation}}{\omega_{orbit}} = \text{integer or } \frac{1}{\text{integer}}$$

look at $n = 1$ ($1/r^2$ force, like gravity)

$$\tilde{r} = \left(\frac{\mu \cdot 1 \cdot A}{\ell^2}\right)^{\frac{1}{1-2}} = \frac{\ell^2}{\mu A}$$

$\ell = \mu r^2 \omega$

$$\omega_{orbit} = \frac{\ell}{\mu \cdot \frac{\ell^4}{\mu^2 A^2}} = \frac{\mu A^2}{\ell^3}$$

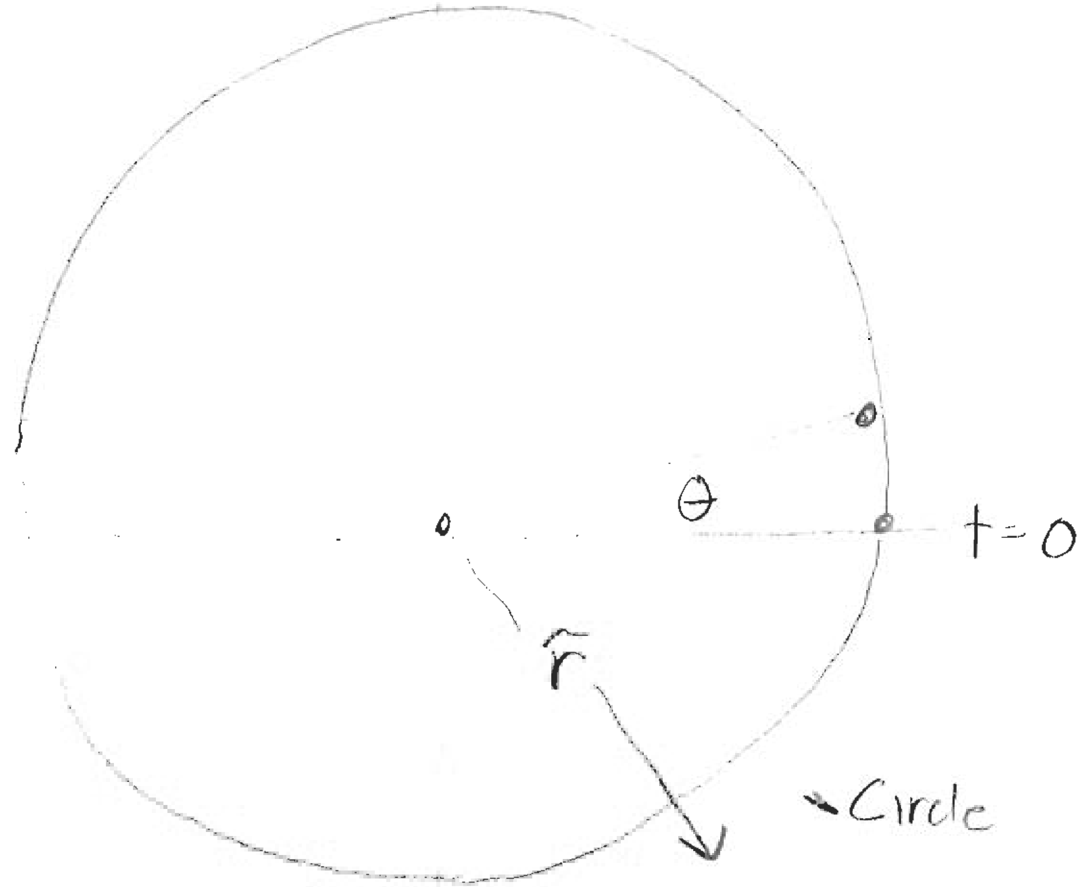
$$\left.\frac{d^2 U_{eff}}{dr^2}\right|_{\tilde{r}} = (2-1)\left(\frac{\ell^2}{\mu}\right)^{\frac{1+2}{1-2}}(1A)^{\frac{-4}{1-2}}$$

$$= \left(\frac{\ell^2}{\mu}\right)^{-3}(1A)^4$$

$$\omega_{oscillation} = \sqrt{\frac{1}{\mu} \cdot \frac{\mu^3}{\ell^6} \cdot A^4}$$

$$\boxed{\omega_{oscillation} = \frac{\mu A^2}{\ell^3}}$$

same as ω_{orbit} !!!



Original Orbit: $\theta = \omega_{orbit} \cdot t$

$r = \hat{r}$

New Orbit (energy added, same angular momentum)

$$r = \hat{r} + B \sin(\omega_{oscillation} \cdot t)$$

$$r = \hat{r} + B \sin\theta \simeq \text{ellipse!}$$