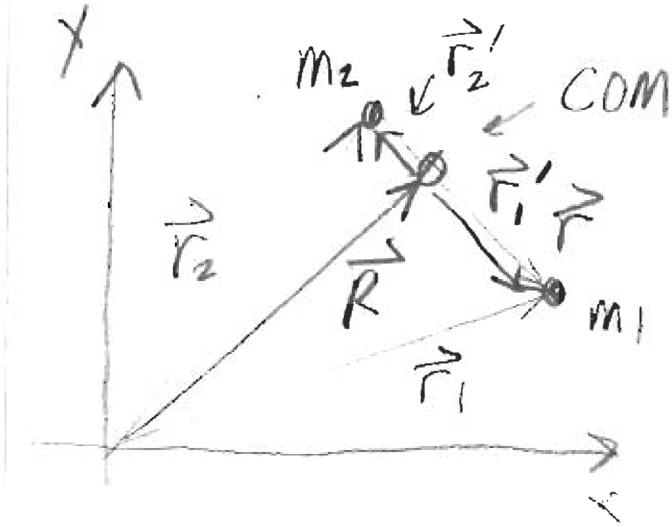
$\mu \ddot{\vec{r}} = f(r)\,\hat{r}$ no $\vec{R}$ involved!



$$\vec{r}_1 = \vec{R} + \underbrace{\frac{m_2}{m_1+m_2}\vec{r}}_{\vec{r}_1{}'}$$

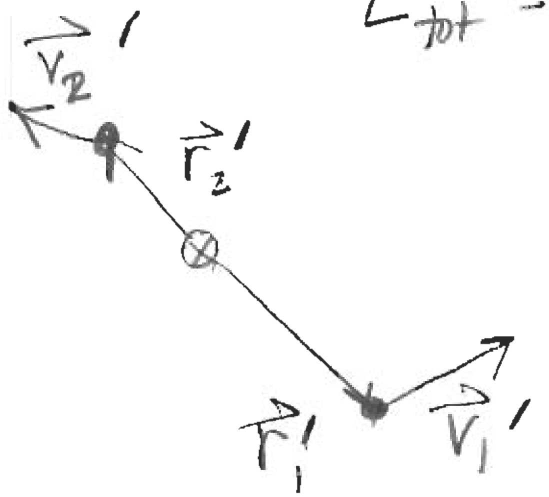
$$\vec{r}_2 = \vec{R} \underbrace{- \frac{m_1}{m_1+m_2}\vec{r}}_{\vec{r}_2{}'}$$ (including − sign)

<u>about</u> COM: $\vec{\tau}_1 = \vec{\tau}_2 = 0$ (force is along $\vec{r}$)

about COM

But, $\vec{L}_1 \neq 0$ nor $\vec{L}_2 \neq 0$

$$\vec{L}_{tot} = \vec{L}_1 + \vec{L}_2$$



$$= m_1 \vec{r}_1{}' \times \vec{v}_1{}' + m_2 \vec{r}_2{}' \times v_2{}'$$

look above

$$= \frac{m_1 m_2}{m_1+m_2}\left(\vec{r}\times\vec{v}_1{}' - \vec{r}\times\vec{v}_2{}'\right)$$

$$= \mu\, \vec{r}\times(\vec{v}_1{}' - \vec{v}_2{}')$$

$$\vec{L}_{tot} = \mu\, \vec{r}\times\vec{v}$$

like a particle mass μ, radius $\vec{r}$! FICTICIOUS EQUIVALENT PARTICLE

$$|\vec{L}| \equiv \ell = \mu r \underbrace{(r\dot{\theta})}_{v_\theta} = \text{constant}$$

$$v_\theta^2 = \frac{\ell^2}{\mu^2 r^2} \qquad \dot{\theta} = \frac{\ell}{\mu r^2}$$

$r(t) \rightarrow \theta(t)$

$$K = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\mu \dot{v}_\theta^2 = \underbrace{\frac{1}{2}\mu \dot{r}^2 + \frac{\ell^2}{2\mu r^2}}_{\text{no } \theta \text{ dependence}}$$

Kinetic

$$U(r) = -\int_{r_0}^{r} f(r)\, dr \quad (r_0 = \text{starting point})$$

= Potential

Energy is conserved

$$E = K + U(r) = \frac{1}{2}\mu \dot{r}^2 + \underbrace{\frac{\ell^2}{2\mu r^2} + U(r)}_{\text{"effective potential"}}$$

$$U_{eff}(r) = \frac{\ell^2}{2\mu r^2} + U(r)$$

$$E = \frac{1}{2}\mu \dot{r}^2 + U_{eff}(r)$$

2 bodies, 3-d $\rightarrow$ 1 body, 3-d $\xrightarrow[\text{force}]{\text{central}}$ 1 body, 1-d

Formal Solution given E

$$\dot{r} = \frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}\left(E - U_{eff}(r)\right)}$$

$$\boxed{\underbrace{\pm\int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - U_{eff}(r)\right)}}}_{\text{just a function of } r} = t - t_0}$$

just a function of r ; invert to give r(t)

start at r_0

$$E = \frac{1}{2}\mu\dot{r}(0)^2 + \frac{\ell^2}{2\mu r_0^2} + U(r_0)$$

→ initial conditions contribute through E determination

→ $\pm$: choose by direction of $\frac{dr}{dt}$ from initial condition

$$\frac{d\theta}{dt} \quad \dot{\theta} = \frac{\ell}{\mu r^2}$$

$$\boxed{\theta - \theta_0 = \int_{t_0}^{t} \frac{\ell}{\mu r^2(t)}\,dt}$$

integrate to $\theta(r)$

$$\frac{d\theta}{dr} = \pm\frac{\ell}{\mu r^2}\frac{1}{\sqrt{\frac{2}{\mu}\left(E - U_{eff}(r)\right)^2}}$$

Summarize 2-body Problem

$$m_1 \ddot{\vec{r}}_1 = f(r)\,\hat{r} \qquad \vec{r} = \vec{r}_1 - \vec{r}_2$$
$$m_2 \ddot{\vec{r}}_2 = -f(r)\,\hat{r} \qquad \text{(no external forces)}$$

"Uncouple" via choice

$$\vec{R} \equiv \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \quad \Rightarrow \vec{R} = \cancel{\vec{R}_1}^{\,0} + \cancel{\vec{V}}^{\,\to 0}\, t$$

choose

$$\vec{r} \equiv \vec{r}_1 - \vec{r}_2$$

then $\mu \ddot{\vec{r}} = f(r)\,\hat{r}$ = "Fictitious 1-body"

$\Rightarrow$ radial force, no torque, orbit in <u>plane</u> $\rightarrow r, \theta$ $\Rightarrow |\vec{L}| \equiv \ell$ constant

$$U(r) \equiv -\int_{r_0}^{r} f(r)\,dr$$

$$E = \frac{1}{2}\mu \dot{r}^2 + \underbrace{\frac{\ell^2}{2\mu r^2} + U(r)}_{U_{eff}(r)} = \text{constant}$$

$$\frac{d\theta}{dr} = \pm \frac{\ell}{\mu r^2} \frac{1}{\sqrt{\frac{2}{\mu}\left(E - U_{eff}(r)\right)}}$$

$\downarrow$ integrate to get $\theta(r)$ $\quad \pm \rightarrow$ different initial `direct...`