

Physics 21 Problem Set 2 – Solutions

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Problem 1 (KK 1.8)

Given vectors

$$\vec{A} = \hat{i} + \hat{j} - \hat{k} \quad \vec{B} = 2\hat{i} - \hat{j} + 3\hat{k}$$

we compute their cross product as follows,

$$\begin{aligned} \vec{A} \times \vec{B} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 2 & -1 & 3 \end{vmatrix} \\ &\text{using cofactor expansion} \\ &= \begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix} \hat{i} - \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} \hat{j} + \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \hat{k} \\ &= 2\hat{i} - 5\hat{j} - 3\hat{k} \end{aligned}$$

A unit normal perpendicular to \vec{A} and \vec{B} is

$$\frac{\vec{A} \times \vec{B}}{\|\vec{A} \times \vec{B}\|} = \frac{2\hat{i} - 5\hat{j} - 3\hat{k}}{\sqrt{38}}$$

Problem 2 (KK 1.11)

We use the following lemma, proved in section:

Lemma 0.1. Either $\vec{A} \parallel \vec{B}$ or \vec{A}, \vec{B} , and $\vec{A} \times \vec{B}$ form a basis.

We apply this lemma to \vec{A}, \hat{n} , and $\vec{A} \times \hat{n}$. It is easy to show

$$\vec{A} = (\vec{A} \cdot \hat{n})\hat{n} + (\hat{n} \times \vec{A}) \times \hat{n}$$

if \vec{A} and \hat{n} are parallel. So we treat the difficult case in which they aren't parallel, but from the lemma, form a basis once we include $\vec{A} \times \hat{n}$. We will show that the equality holds in each of the three components formed by our basis, so as a consequence the equation is true as a vector equation. In the \vec{A} direction,

$$\begin{aligned} \vec{A}^2 &= (\vec{A} \cdot \hat{n})^2 + (\hat{n} \times \vec{A})^2 \\ &= A^2 \cos^2 \theta + A^2 \sin^2 \theta \\ &= A^2 \end{aligned}$$

In the \hat{n} direction,

$$\begin{aligned}\vec{A} \cdot \hat{n} &= (\vec{A} \cdot \hat{n})\hat{n}^2 + ((\hat{n} \times \vec{A}) \times \hat{n}) \cdot \hat{n} \\ &\quad \text{since } (\hat{n} \times \vec{A}) \times \hat{n} \text{ is perpendicular to } \hat{n} \\ &= \vec{A} \cdot \hat{n}\end{aligned}$$

In the $\vec{A} \times \hat{n}$ direction,

$$\vec{A} \cdot (\vec{A} \times \hat{n}) = 0,$$

so we must show

$$\left((\vec{A} \cdot \hat{n})\hat{n} + (\hat{n} \times \vec{A}) \times \hat{n} \right) \cdot (\vec{A} \times \hat{n}) = 0.$$

This is true since both pieces are individually 0.

$$(\vec{A} \cdot \hat{n})\hat{n} \cdot (\vec{A} \times \hat{n}) = 0$$

since $\vec{A} \times \hat{n}$ is perpendicular to \hat{n} . Similarly

$$\left((\hat{n} \times \vec{A}) \times \hat{n} \right) \cdot (\vec{A} \times \hat{n}) = 0$$

since $\vec{A} \times \hat{n}$ appears twice in the triple-cross product.

(a)

The unit normal vector \hat{b} in the direction of \vec{B} is

$$\hat{b} = \frac{2\hat{i} - \hat{j} + 3\hat{k}}{\sqrt{14}}$$

(b)

The component of \vec{A} parallel to \vec{B} is

$$\begin{aligned}\vec{V}_1 &= (\vec{A} \cdot \hat{b})\hat{b} \\ &= \frac{(\vec{A} \cdot \vec{B})\vec{B}}{\|\vec{B}\|^2} \\ &= -\frac{\vec{B}}{7} \\ &= \frac{1}{7}(-2\hat{i} + \hat{j} - 3\hat{k})\end{aligned}$$

(c)

The component of \vec{A} perpendicular to \vec{B} is

$$\begin{aligned}\vec{V}_2 &= (\hat{b} \times \vec{A}) \times \hat{b} \\ &= \frac{1}{\|\vec{B}\|^2}(\vec{B} \times \vec{A}) \times \vec{B} \\ &= -\frac{1}{14}(2\hat{i} - 5\hat{j} - 3\hat{k}) \times (2\hat{i} - \hat{j} + 3\hat{k}) \\ &= \frac{1}{7}(9\hat{i} + 6\hat{j} - 4\hat{k})\end{aligned}$$

(d)

$$\begin{aligned}\vec{V}_1 + \vec{V}_2 &= \frac{1}{7}(-2\hat{i} + \hat{j} - 3\hat{k}) + \frac{1}{7}(9\hat{i} + 6\hat{j} - 4\hat{k}) \\ &= \hat{i} + \hat{j} - \hat{k}\end{aligned}$$

so $\vec{V}_1 + \vec{V}_2 = \vec{A}$ as expected, since \vec{V}_1 and \vec{V}_2 are the components of \vec{A} parallel and perpendicular to \vec{B} .

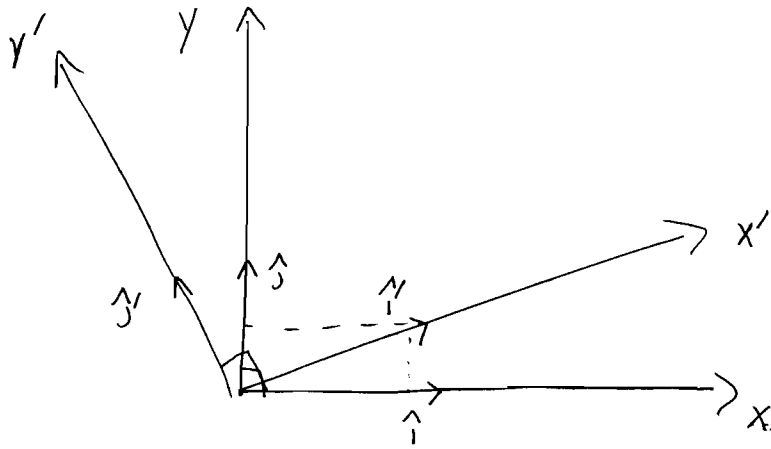
(e)

$$\begin{aligned}\vec{B} \cdot \vec{V}_2 &= \frac{1}{7}(2\hat{i} - \hat{j} + 3\hat{k}) \cdot (-2\hat{i} + \hat{j} - 3\hat{k}) \\ &= \frac{1}{7}(18 - 6 - 12) \\ &= 0\end{aligned}$$

so \vec{V}_2 is perpendicular to \vec{B} as expected.

(3)

(a)



$$\begin{cases} \hat{i}' = \cos 30^\circ \hat{i} + \sin 30^\circ \hat{j} \\ \hat{j}' = -\sin 30^\circ \hat{i} + \cos 30^\circ \hat{j} \end{cases}$$

$$\begin{pmatrix} \hat{i}' \\ \hat{j}' \end{pmatrix} = R(\theta) \begin{pmatrix} \hat{i} \\ \hat{j} \end{pmatrix}$$

(b) $\hat{i} \cdot \hat{i}' = \cos 30^\circ = \frac{\sqrt{3}}{2}$

(c) $\hat{i} \cdot \hat{j}' = -\sin 30^\circ = -\frac{1}{2}$

(d) $\hat{j} \cdot \hat{i}' = \sin 30^\circ = \frac{1}{2}$

(e) $\hat{j} \cdot \hat{j}' = \cos 30^\circ = \frac{\sqrt{3}}{2}$

(f) To express \hat{i} and \hat{j} in terms of \hat{i}' and \hat{j}' we must invert

(g) the relation matrix

$$\begin{pmatrix} \hat{i} \\ \hat{j} \end{pmatrix} = R^{-1}(\theta) \begin{pmatrix} \hat{i}' \\ \hat{j}' \end{pmatrix} = R(-\theta) \begin{pmatrix} \hat{i}' \\ \hat{j}' \end{pmatrix}$$

$$\hat{i} = \cos 30^\circ \hat{i}' - \sin 30^\circ \hat{j}'$$

$$\hat{j} = +\sin 30^\circ \hat{i}' + \cos 30^\circ \hat{j}'$$

$$\hat{i} = \frac{\sqrt{3}}{2} \hat{i}' - \frac{1}{2} \hat{j}'$$

$$\hat{j} = \frac{1}{2} \hat{i}' + \frac{\sqrt{3}}{2} \hat{j}'$$

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Problem 4

A particle moves in one dimension with a position described by

$$x(t) = t - t^3/3! + t^5/5!$$

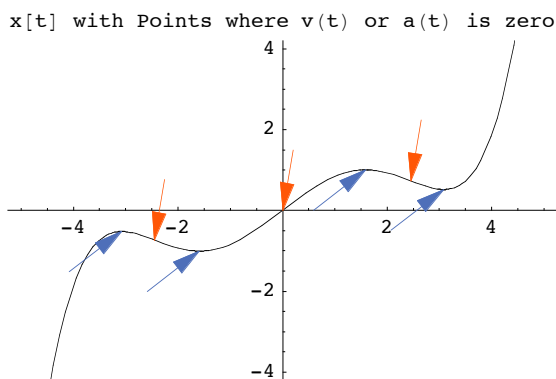
where x is in meters and t is in seconds.

■ Initialization Code

■ (a)

Plot of $x(t)$ with critical points marked

```
In[37]:= Plot[x[t], {t, -5, 5}, PlotLabel -> "x[t] with Points where v(t) or a(t) is zero",
  Epilog -> {Cobalt, Table[Arrow[vex[[i]] - {1, 1}, vex[[i]]], {i, 1, Length[vex]}],
  {GreenishUmbre, Table[Arrow[acx[[i]] + {.2, 1.5}, acx[[i]]], {i, 1, Length[acx]}]}}
```



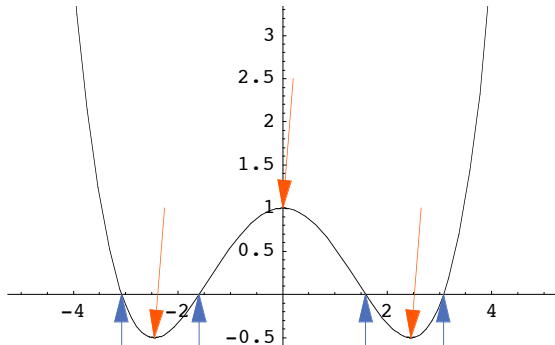
Out[37]= - Graphics -

■ (b)

Plot of $v(t)$ with critical points marked

```
In[38]:= Plot[Evaluate[D[x[t], t]], {t, -5, 5},
  PlotLabel -> "v[t] with Points where v(t) or a(t) is zero",
  Epilog -> {Cobalt, Table[Arrow[vcv[[i]] - {0, 1}, vcv[[i]]], {i, 1, Length[vcv]}],
  {GreenishUmer, Table[Arrow[acv[[i]] + {.2, 1.5}, acv[[i]]], {i, 1, Length[acv]}]}}
```

v[t] with Points where v(t) or a(t) is zero



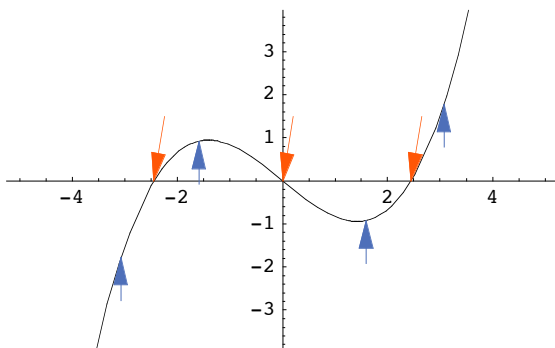
Out[38]= - Graphics -

■ (c)

Plot of a(t) with critical points marked

```
In[39]:= Plot[Evaluate[D[x[t], {t, 2}]], {t, -5, 5},
  PlotLabel -> "a[t] with Points where v(t) or a(t) is zero",
  Epilog -> {Cobalt, Table[Arrow[vca[[i]] - {0, 1}, vca[[i]]], {i, 1, Length[vca]}],
  {GreenishUmer, Table[Arrow[aca[[i]] + {.2, 1.5}, aca[[i]]], {i, 1, Length[aca]}]}}
```

a[t] with Points where v(t) or a(t) is zero

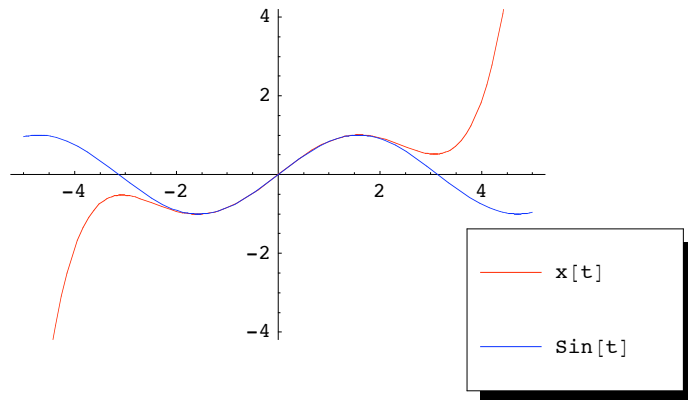


Out[39]= - Graphics -

■ (d)

Plot sin(t) along with its Taylor Approximation

```
In[36]:= Plot[{x[t], Sin[t]}, {t, -5, 5}, PlotStyle -> {Red, Blue},
  LegendPosition -> {.7, -.8}, PlotLegend -> {"x[t]", "Sin[t]"}]
```



```
Out[36]= - Graphics -
```

Problem 5

A particle moves in an elliptical path, centered on the origin and with semi-major axis $a = 1\text{m}$ parallel to x , and semi-minor axis $b = 0.2\text{m}$ parallel to y . The particle moves with constant angular velocity, so that its period is $T = 2\text{s}$.

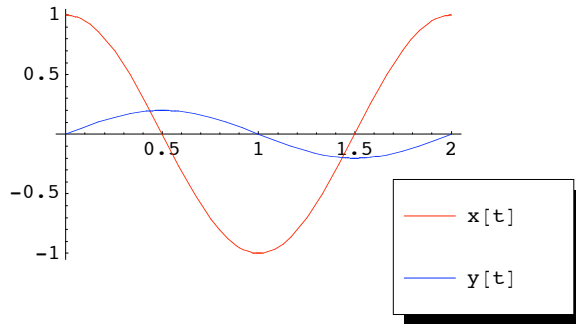
```
In[46]:= a = 1;
  b = 0.2;
  T = 2;
  ω = 2 * Pi / T;
```

```
In[52]:= x[t_] := a Cos[ω * t]
  y[t_] := b * Sin[ω * t]
```

■ (a)

Evaluate $x(t)$ and $y(t)$ symbolically and numerically. Plot them, with the same time axis and with a common t -axis, for t from 0 to two periods, but with x 's origin displaced vertically from that of y 's.

```
In[54]:= Plot[{x[t], y[t]}, {t, 0, 2}, PlotStyle -> {Red, Blue},
  LegendPosition -> {.7, -.8}, PlotLegend -> {"x[t]", "y[t]"}]
```



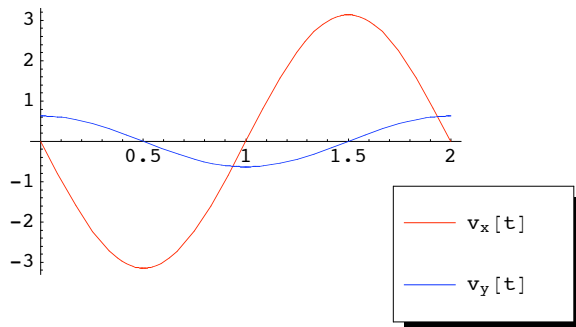
Out[54]= - Graphics -

■ (b)

Repeat the last step for $v_x(t)$ and $v_y(t)$

```
In[69]:= vx[t_] = D[x[t], t];
  vy[t_] = D[y[t], t];
```

```
In[83]:= Plot[Evaluate[{vx[t], vy[t]}], {t, 0, 2}, PlotStyle -> {Red, Blue},
  LegendPosition -> {.7, -.8}, PlotLegend -> {"v_x[t]", "v_y[t]"}]
```



Out[83]= - Graphics -

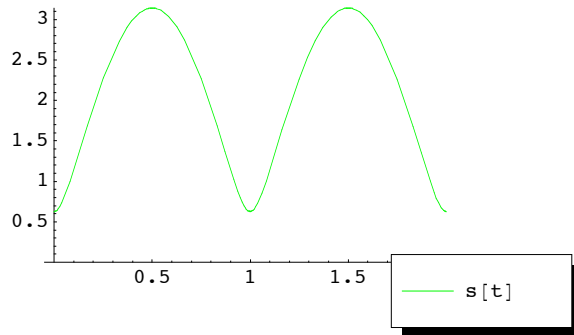
■ (c)

Plot the speed as a function of t:

Speed is given by

```
In[75]:= s[t_] = Sqrt[vx[t]^2 + vy[t]^2];
```

```
In[77]:= Plot[s[t], {t, 0, 2}, PlotStyle -> {Green},
  LegendPosition -> {.7, -.8}, PlotLegend -> {"s[t]"}, PlotRange -> {0,  $\pi$ }]
```



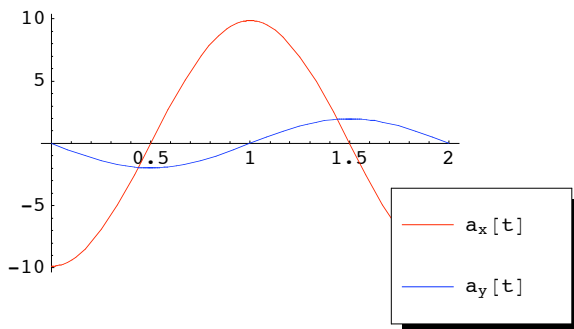
Out[77]= - Graphics -

■ (d)

Repeat part (b) for $a_x(t)$ and $a_y(t)$

```
In[79]:= ax[t_] = D[x[t], {t, 2}];
  ay[t_] = D[y[t], {t, 2}];
```

```
In[82]:= Plot[Evaluate[{ax[t], ay[t]}], {t, 0, 2}, PlotStyle -> {Red, Blue},
  LegendPosition -> {.7, -.8}, PlotLegend -> {"a_x[t]", "a_y[t]"}]
```



Out[82]= - Graphics -