# De Moivre's Theorem

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \ldots$$

$$e^{i\theta} = 1 + i\theta + \frac{1}{2!}(i\theta)^{2} + \frac{1}{3!}(i\theta)^{3} + \frac{1}{4!}(i\theta)^{4} + \ldots$$

$$\cos\theta = 1 \qquad - \frac{1}{2!}\theta^{2} \qquad\qquad + \frac{1}{4!}\theta^{4} \ldots$$

$$i\sin\theta = i( \quad \theta \qquad\qquad - \frac{1}{3!}\theta^{3} \qquad \ldots$$

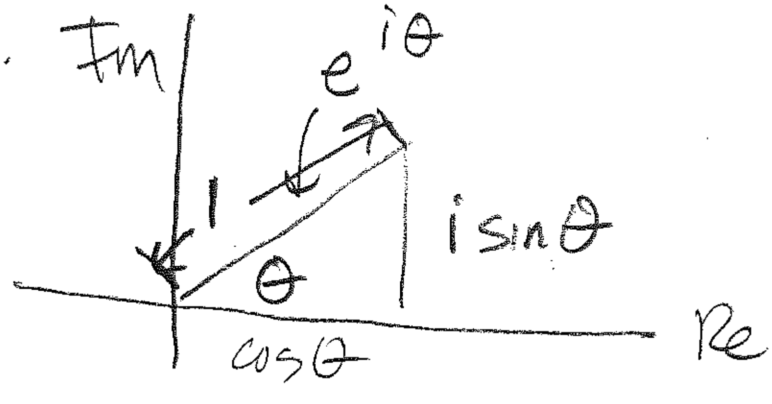
COMPARE ...

$$|e^{i\theta}| = \sqrt{(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}$$

$$= \sqrt{\cos^{2}\theta - \cancel{i\cos\theta\sin\theta} + \cancel{i\sin\theta\cos\theta} + \sin^{2}\theta}$$

$$= \sqrt{\cos^{2}\theta + \sin^{2}\theta} = 1 \quad !$$

Plot ...



any number $z$ ...

$$z = x + iy = \sqrt{x^2+y^2}\, e^{i\theta} = re^{i\theta}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad \sqrt{x^2+y^2}$$

$$z^* = x - iy = \sqrt{x^2+y^2}\, e^{-i\theta} = re^{-i\theta}$$

$$2 - 2i =$$

(A) $2e^{i\pi/2}$

(B) $2e^{-i\pi/4}$

(C) $2e^{i\pi/4}$

(D) $2e^{-i\pi/2}$

Complex Conjuate of

$e^{i\theta}$ when $\theta$ is real

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$(e^{i\theta})^* = (\cos\theta + i\sin\theta)^*$$

$$= \cos\theta - i\sin\theta$$

$$= e^{-i\theta}$$

so, $(e^{i\theta})(e^{i\theta})^*$

$$= e^{i\theta} e^{-i\theta}$$

$$= e^{i(\theta-\theta)} = e^{i0}$$

$$= 1$$

# Derivative of $e^{i\theta}$

$$\frac{d}{d\theta}\left(e^{i\theta}\right) = \frac{d}{d\theta}\left(\cos\theta + i\sin\theta\right)$$

$$e^{i\theta} \times i \overset{\text{check}}{=} -\sin\theta + i\cos\theta$$

$$(\cos\theta + i\sin\theta) \times i \overset{?}{=} -\sin\theta + i\cos\theta$$

$$i \qquad -1$$

$$i\cos\theta - \sin\theta \overset{?}{=} -\sin\theta + i\cos\theta \quad \checkmark$$

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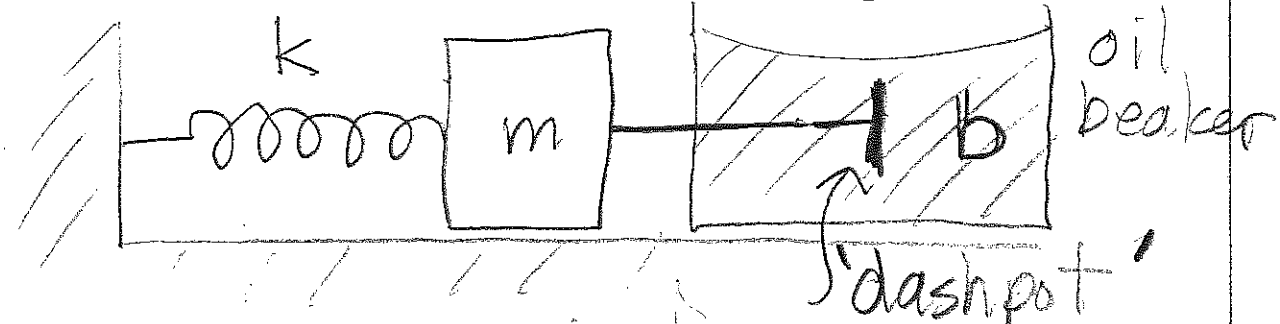
# Damped simple harmonic oscillator

model for many, many "damped" systems

→ car / bike shocks

→ musical instruments

→ electromagnetic wave generators




| | |
|---|---|
| $b = 0$ no damping | $b > 0$ damping |
| $m\ddot{x} = -kx$ | $m\ddot{x} = -kx \underbrace{- b\dot{x}}_{\text{... opposes velocity}}$ |
| let $x(t)$ satisfy above, + $y(t)$ <u>also</u> | |

satisfy it. Then $z(t) \equiv x(t) + iy(t)$,
$z(t)$ is complex... use it to solve
$m\ddot{z} = -kz$, and then use real part to describe actual motion. Arbitrary... could take $\mathrm{Im}(z)$.

Try $z(t) = z_0 e^{\alpha t}$ $\boxed{\begin{array}{l}\alpha \to \text{complex} \\ \text{IMPORTANT}\end{array}}$

(initial conditions get this

$$\dot{z} = z_0 \alpha e^{\alpha t}$$

$$\ddot{z} = z \ \alpha^2 e^{\alpha t}$$

$$m \cancel{z_0} \alpha^2 \cancel{e^{\alpha t}} = -k \cancel{z_0} \cancel{e^{\alpha t}}$$

$$\alpha^2 = -\frac{k}{m} = -\omega_0^2$$ "natural circular frequency"

$$\alpha_{1,2} = \pm\sqrt{-\omega_0^2}$$

$$\alpha_{1,2} = \pm i\omega_0$$ ] TWO SOLUTIONS

THIS CRUCIAL, WILL PERSIST

$\boxed{\text{GENERAL SOLUTION}}$

$$z(t) = z_A e^{\alpha_1 t} + z_B e^{\alpha_2 t}$$

$$= \underbrace{z_A e^{i\omega_0 t} + z_B e^{-i\omega_0 t}}_{\text{want real part, means}}$$

real

$$z_A = \tfrac{1}{2} X_m e^{i\phi}$$

$$z_B = \tfrac{1}{2} X_m e^{-i\phi}$$

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has all the info ... no loss of generality

and $z(t) = \frac{1}{2} x_m \left( e^{i\omega_0 t + i\phi} + e^{-i\omega_0 t - i\phi} \right)$

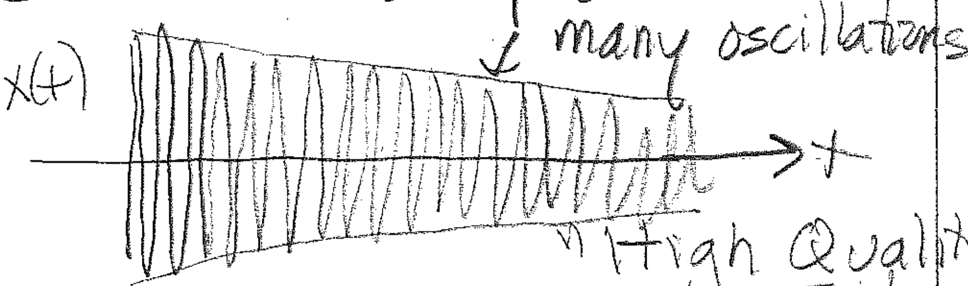
$= x_m \cos(\omega_0 t + \phi)$

NOW make $b \neq 0$, $b > 0$ "DAMPING"

Important: frequencies in exponent now solutions to the quadratic equation (12 year-olds sing solution!)

Physical character will depend on the discriminant ...

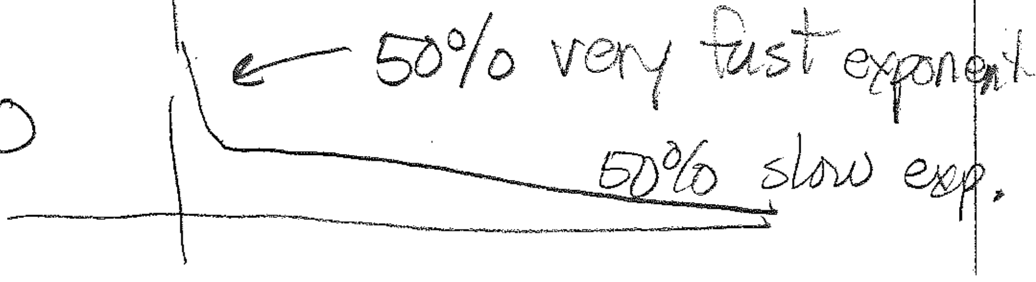
$D < 0$ "underdamped"




"High Quality"

$D = 0$ "critically damped"


$D > 0$



We'll focus on underdamped...
a little on critically damped.

$$m\ddot{z} = -kz - b\dot{z} \qquad z = z_0 e^{\alpha t}$$

$$m \cancel{z_0} \alpha^2 \cancel{e^{\alpha t}} = -k\cancel{z_0} \cancel{e^{\alpha t}} - b\cancel{z_0}\alpha \cancel{e^{\alpha t}}$$

$$m\alpha^2 = -k - b\alpha$$ ← divide through by m, rearrange

$$\alpha^2 + \frac{b}{m}\alpha + \frac{k}{m} = 0$$

call $\gamma$ ($1/\text{time}$), call $\omega_0^2$, dimensions $1/\text{time}^2$

$\gamma \to 0$ no damping.

$$\alpha^2 + \gamma\alpha + \omega_0^2 = 0$$ THE QUADRATIC.

$$D \text{ "}=\text{" } \sqrt{b^2 - 4ac} \text{ "}=\text{" } 2\sqrt{\left(\frac{b}{2}\right)^2 - ac}$$

$$= 2\sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2}$$

$\boxed{\text{LOOK! } \gamma \to 0 \quad = 2i\omega_0}$

UNDERDAMPED $\left(\frac{\gamma}{2}\right)^2 - \omega_0^2 < 0$

CRITICALLY DAMPED $\left(\frac{\gamma}{2}\right) - \omega_0^2 = 0$

OVERDAMPED $\left(\frac{\gamma}{2}\right) - \omega_0^2 > 0$

→ solution pretty similar to $x_m \cos(\omega_0 t + \phi)$

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$$\alpha_{1,2} = \frac{-\gamma \pm 2\sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2}}{2}$$

definition

$$= -\frac{\gamma}{2} \pm \sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2} \qquad -\omega_1^2 \equiv \left(\frac{\gamma}{2}\right)^2 - \omega_0^2$$

$$z(t) = \frac{1}{2} x_m \left[ e^{-\frac{\gamma}{2}t} e^{i\omega_1 t + i\phi} + e^{-\frac{\gamma}{2}t} e^{-i\omega_1 t - i\phi} \right]$$

COMMON!

$$\boxed{x(t) = x_m e^{-\frac{\gamma}{2}t} \cos(\omega_1 t + \phi)} \quad \underline{\text{REAL}}$$

$\boxed{\underline{\text{Principal}} \text{ New Physical Effect}}$ — $\boxed{e^{-\frac{\gamma}{2}t}}$

Secondary: $\omega_1 \neq \omega_0$, but, shift is "second order"

$x_m, \phi \rightarrow$ • initial conditions
• release from rest, $\phi = 0$,
$x_m$ = initial displacement

$\boxed{\text{Energy}}$ $= \frac{1}{2} k x^2(t) + \frac{1}{2} m \dot{x}^2(t)$

$$\dot{x}(t) = x_m \left[ \underbrace{-\frac{\gamma}{2} e^{-\gamma t/2} \cos(\overset{\simeq \omega_0}{\omega_1} t + \phi)}_{\text{limit } \gamma \to 0, \text{ neglect}} - \underset{\simeq \omega_0}{\omega_1} e^{-\gamma t/2} \sin(\underset{\simeq \omega_0}{\omega_1} t + \phi) \right]$$

$$\dot{x}(t) \simeq -x_m \omega_0 e^{-\gamma t/2} \sin(\omega_0 t + \phi)$$

$$E = \frac{1}{2} k x_m^2 e^{-\gamma t} \cos^2(\omega_0 t + \phi) + \frac{1}{2} m x_m^2 \omega_0^2 e^{-\gamma t} \sin^2(\omega_0 t + \phi)$$

$\simeq \omega_1$ $\qquad$ $m\omega_0^2 = k$

$$= \frac{1}{2} k x_m^2 e^{-\gamma t} \left( \underbrace{\cos^2(\omega_0 t + \phi) + \sin^2(\omega_0 t + \phi)}_{1} \right)$$

$$\boxed{E \simeq \frac{1}{2} k x_m^2 e^{-\gamma t}}$$

ENERGY PURELY EXPONENTIALLY DECAYS

square of max $\qquad$ $\gamma/2$ gets squard to $\gamma$

**Quality Factor $Q$**

fractional energy loss/radian

$\equiv 1/Q$ (dimensionless).

$$Q = \frac{E}{-\Delta E} = \frac{\cancel{\frac{1}{2} k x_m^2 e^{-\gamma t}}}{-\left(\frac{1}{\omega_0}\right) \cdot \left(-\cancel{\frac{1}{2} k x_m^2}\, \gamma \cancel{e^{-\gamma t}}\right)}$$

$$Q = \frac{\omega_0}{\gamma} \quad \text{DIMENSIONLESS}$$

$Q \to \infty$ then $\gamma \to 0$. NO DAMPING

"high quality"

## $\boxed{\text{Intuition on } Q}$ ... (can't do demo!)

Lots of Systems use $Q$ as a measure of "Quality" = absence of damping

① Watch until peak amplitude falls to $\simeq e^{-\pi} \cong 4\% \simeq \frac{1}{23}$ of original (LOTS of Cycles!)

② Count $\nu$, # cycles.. $\boxed{Q = \nu!}$

$$\nu = \frac{\omega_1 t}{2\pi} \simeq \frac{\omega_0 t}{2\pi}$$

$$\text{so} \quad t = \frac{2\pi\nu}{\omega_0}$$

$$e^{-\frac{\gamma t}{2}} = e^{-\pi}$$

$$\text{so} \quad \gamma \cdot \frac{\cancel{2}\cancel{\pi}\nu}{\cancel{2}\omega_0} = \cancel{\pi}$$

$$\boxed{\nu = \frac{\omega_0}{\gamma} = Q}$$

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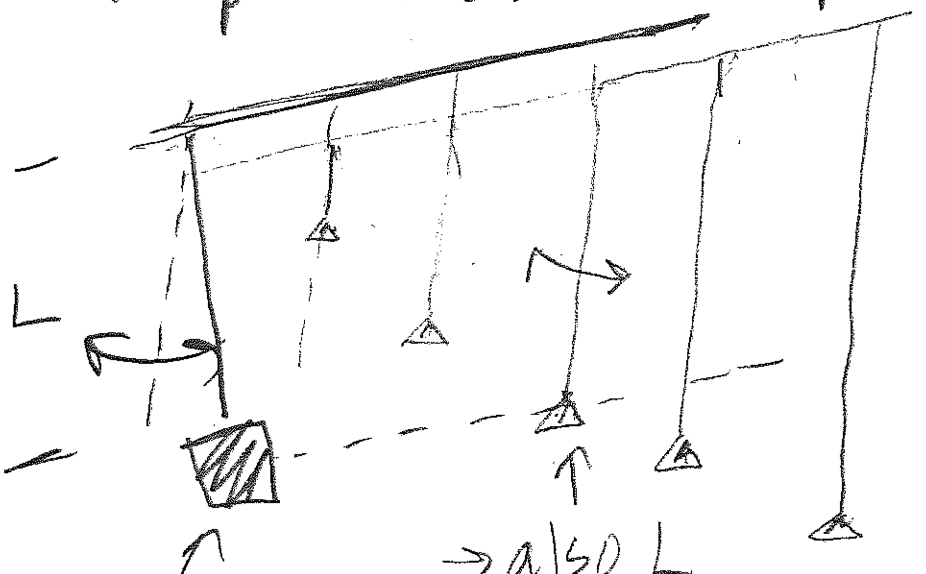
# Forced, Damped Oscillator.

$\omega \rightarrow$ driving circular frequency

Example was multiple pendula...

(Demo 40,42)




big mass

$\omega$

= driving frequency (or period)

$$\omega = \sqrt{\frac{g}{L}}$$

(all move at $\omega$ !) just poorly

"natural frequency"

$\omega_0 \simeq \omega$

"at resonance."

$\rightarrow$ phase shifted by $-i$

$\rightarrow$ biggest amplitude.

Generally ...

$$\underset{\substack{\uparrow \\ \text{drivin}}}{m\ddot{x}} = \underbrace{-kx}_{\substack{\text{spring} \\ \text{behavior} \\ \text{of driven} \\ \text{system}}} \underbrace{-b\dot{x}}_{\text{damping}} + \underset{\substack{\uparrow \\ \underline{\underline{\text{driver}}}}}{F_0 \cos(\omega t)}$$

$\mathrm{Re}(e^{i\omega t}) = \cos(\omega t)$

$x = \mathrm{Re}(z)$ as before

then $\ddot{z} = -\frac{k}{m} z - \frac{b}{m}\dot{z} + \frac{F_0}{m} e^{i\omega t}$

natural freq

$$= -\omega_0^2 z - \gamma \dot{z} + \frac{F_0}{m} e^{i\omega t}$$

$\boxed{\text{IN STEADY STATE}}$ → large times

$z = z_0 e^{i\omega t}$ ← cancels everywhere

$\dot{z} = i\omega z_0 e^{i\omega t}$

$\ddot{z} = -\omega^2 z_0 e^{i\omega t}$

$$-\omega^2 z_0 \cancel{e^{i\omega t}} = -\omega_0^2 z_0 \cancel{e^{i\omega t}} - i\gamma\omega z_0 \cancel{e^{i\omega t}} + \frac{F_0}{m} \cancel{e^{i\omega t}}$$

This time, $z_0$ doesn't cancel, but

$$z_0(\omega_0^2 - \omega^2 + i\gamma\omega) = \frac{F_0}{m}$$

$$z_0 = \frac{F_0}{m(\omega_0^2 - \omega^2 + i\gamma\omega)}$$

$$= \frac{F_0}{\underbrace{m\omega_0^2}_{m\cdot\frac{k}{m} = k} \underbrace{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2 + i\frac{\gamma\omega}{\omega_0^2}\right)}_{\text{dimensionless}}}$$

$$z_0 = \frac{F_0}{k} \cdot \underbrace{\frac{1}{\left[1 - \left(\frac{\omega}{\omega_0}\right)^2 + i\frac{\gamma\omega}{\omega_0^2}\right]}}_{\text{Resonant Factor}}$$

DISPLACEMENT CAUSED BY $F_0$ IF STATIC

Resonant Factor
Imagine this as a factor of $\omega$

then $x(t) = \frac{F_0}{k} \mathrm{Re}\left[\frac{1}{\left[1 - \left(\frac{\omega}{\omega_0}\right)^2 + i\frac{\gamma\omega}{\omega_0^2}\right]} e^{i\omega t}\right]$

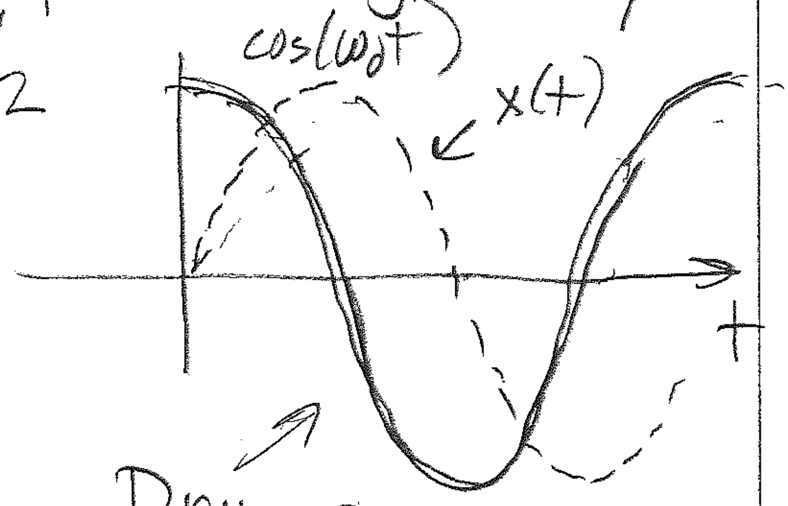
(A) $\omega = \omega_0$ RESONANCE

$$x(t) = \frac{F_0}{k} \mathrm{Re}\left[-i \frac{\omega_0}{\gamma} e^{i\omega_0 t}\right]$$

STEADY STATE THE $-i$

$$= \underbrace{\frac{F_0}{m} Q \sin(\omega_0 t)}$$

- DRIVE WITH $\cos(\omega_0 t)$
- Responce ↑ by factor of $Q$ = Quality.
- Response Lags by... $\pi/2$

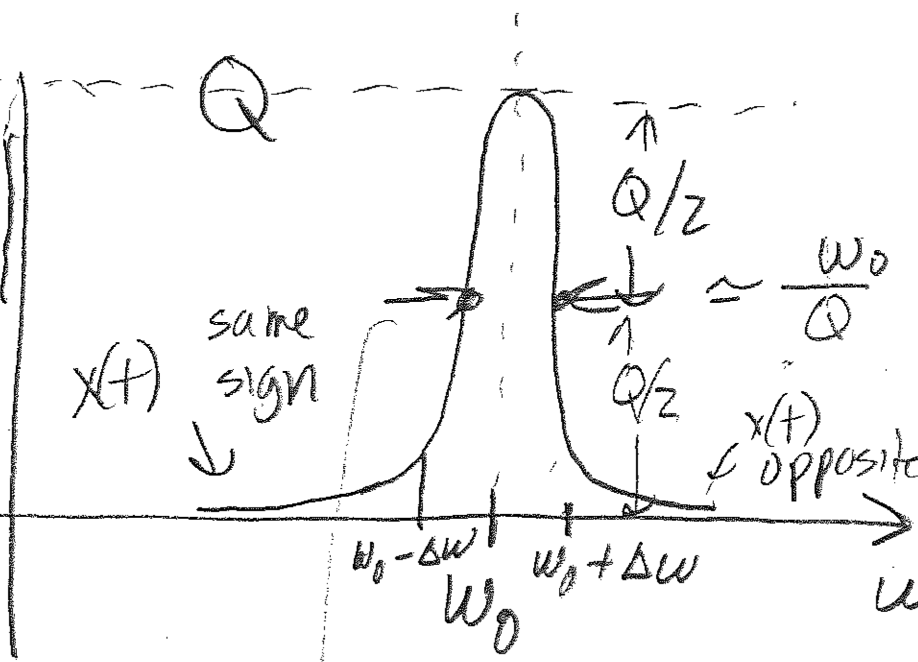



(B) $\omega < \omega_0$ $x(t)$ & $F_0 \cos(\omega t)$ SAME SIGN, $x(t)$ small

(c) $\omega > \omega_0$ $x(t)$ + $F_0 \cos(\omega t)$
OPPOSITE SIGN.

"Resonance Curve"

$$\left|\frac{1}{1-\left(\frac{\omega}{\omega_0}\right)^2 + i\frac{\gamma\omega}{\omega_0^2}}\right|$$




"Width" $\rightarrow$ FWHM $\simeq \omega_0/Q$
Full Width Half Max

$$1-\left(\frac{\omega_0 \pm \Delta\omega}{\omega_0}\right)^2 \simeq 1 - 1 \mp 2\frac{\omega_0 \Delta\omega}{\omega_0^2} + O(\Delta\omega^2)$$

$$\simeq \mp 2\frac{\Delta\omega}{\omega_0}$$

$$\frac{\gamma\omega}{\omega_0^2} \simeq \frac{\gamma}{\omega_0} = \frac{1}{Q} \qquad \omega = \omega_0 \pm \Delta\omega$$

$$\frac{1}{\left|\mp 2\frac{\Delta\omega}{\omega_0} + \frac{i}{Q}\right|} = \frac{1}{\sqrt{4\left(\frac{\Delta\omega}{\omega}\right)^2 + \frac{1}{Q^2}}} = \frac{Q}{2}$$

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$$4\left(\frac{\Delta\omega}{\omega_0}\right)^2 + \frac{1}{Q^2} = \frac{4}{Q^2}$$

$$\frac{\Delta\omega}{\omega_0} = \sqrt{\frac{3}{4}}\,\frac{1}{Q}$$

$$\boxed{\Delta\omega = \sqrt{\frac{3}{4}}\,\frac{\omega_0}{Q}} \simeq \frac{\omega_0}{Q}$$

Key point: to see maximum resonant effect, must tune $\omega$ to within $\sim 1/Q$ of $\omega_0$ (%-wise).

Disclaimer: for energy, not amplitude,

$$\Delta\omega = \frac{\omega_0}{Q} \;---$$

$\sqrt{\frac{3}{4}}$ comes from amplitude.

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