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AN INTRODUCTION TO MECHANICS

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CONTENTS

1 VECTORS AND KINEMATICS— A FEW MATHEMATICAL PRELIMINARIES

1.1 Introduction

The goal of this book is to help you acquire a deep understanding of the principles of mechanics. The subject of mechanics is at the very heart of physics; its concepts are essential for understanding the everyday physical world as well as phenomena on the atomic and cosmic scales. The concepts of mechanics, such as momentum, angular momentum, and energy, play a vital role in practically every area of physics.

We shall use mathematics frequently in our discussion of physical principles, since mathematics lets us express complicated ideas quickly and transparently, and it often points the way to new insights. Furthermore, the interplay of theory and experiment in physics is based on quantitative prediction and measurement. For these reasons, we shall devote this chapter to developing some necessary mathematical tools and postpone our discussion of the principles of mechanics until Chap. 2.

1.2 Vectors

The study of vectors provides a good introduction to the role of mathematics in physics. By using vector notation, physical laws can often be written in compact and simple form. (As a matter of fact, modern vector notation was invented by a physicist, Willard Gibbs of Yale University, primarily to simplify the appearance of equations.) For example, here is how Newton's second law (which we shall discuss in the next chapter) appears in nineteenth century notation:

$$F_x = ma_x$$
$$F_y = ma_y$$
$$F_z = ma_z.$$

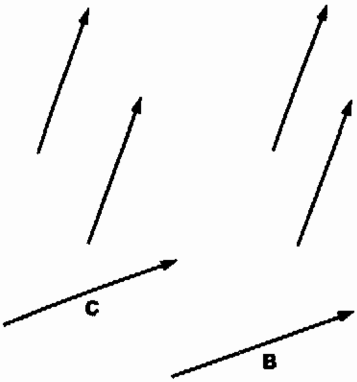
In vector notation, one simply writes

$$\mathbf{F} = m\mathbf{a}.$$

Our principal motivation for introducing vectors is to simplify the form of equations. However, as we shall see in the last chapter of the book, vectors have a much deeper significance. Vectors are closely related to the fundamental ideas of symmetry and their use can lead to valuable insights into the possible forms of unknown laws.

Definition of a Vector

Vectors can be approached from three points of view—geometric, analytic, and axiomatic. Although all three points of view are useful, we shall need only the geometric and analytic approaches in our discussion of mechanics.



From the geometric point of view, a vector is a *directed line segment.* In writing, we can represent a vector by an arrow and label it with a letter capped by a symbolic arrow. In print, boldfaced letters are traditionally used.

In order to describe a vector we must specify both its length and its direction. Unless indicated otherwise, we shall assume that parallel translation does not change a vector. Thus the arrows at left all represent the same vector.

If two vectors have the same length and the same direction they are equal. The vectors **B** and **C** are equal:

$$\mathbf{B} = \mathbf{C}.$$

The length of a vector is called its *magnitude.* The magnitude of a vector is indicated by vertical bars or, if no confusion will occur, by using italics. For example, the magnitude of **A** is written $|\mathbf{A}|$, or simply A. If the length of **A** is $\sqrt{2}$, then $|\mathbf{A}| = A = \sqrt{2}$.

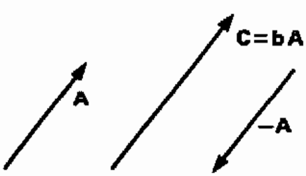
If the length of a vector is one unit, we call it a *unit vector.* A unit vector is labeled by a caret; the vector of unit length parallel to **A** is $\hat{\mathbf{A}}$. It follows that

$$\hat{\mathbf{A}} = \frac{\mathbf{A}}{|\mathbf{A}|},$$

and conversely

$$\mathbf{A} = |\mathbf{A}|\hat{\mathbf{A}}.$$

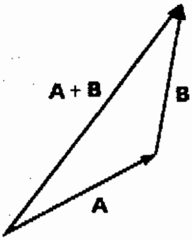
The Algebra of Vectors



Multiplication of a Vector by a Scalar If we multiply **A** by a positive scalar b, the result is a new vector $\mathbf{C} = b\mathbf{A}$. The vector **C** is parallel to **A**, and its length is b times greater. Thus $\hat{\mathbf{C}} = \hat{\mathbf{A}}$, and $|\mathbf{C}| = b|\mathbf{A}|$.

The result of multiplying a vector by -1 is a new vector opposite in direction (antiparallel) to the original vector.

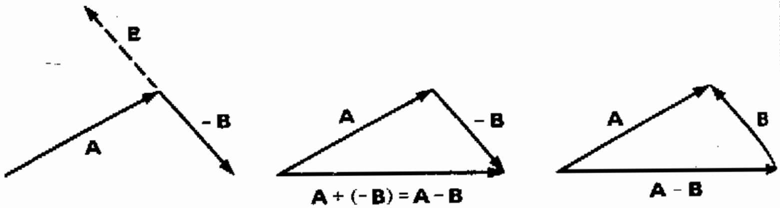
Multiplication of a vector by a negative scalar evidently can change both the magnitude and the direction sense.



Addition of Two Vectors Addition of vectors has the simple geometrical interpretation shown by the drawing.

The rule is: To add **B** to **A**, place the tail of **B** at the head of **A**. The sum is a vector from the tail of **A** to the head of **B**.

Subtraction of Two Vectors Since $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$, in order to subtract **B** from **A** we can simply multiply it by -1 and then add. The sketches below show how.

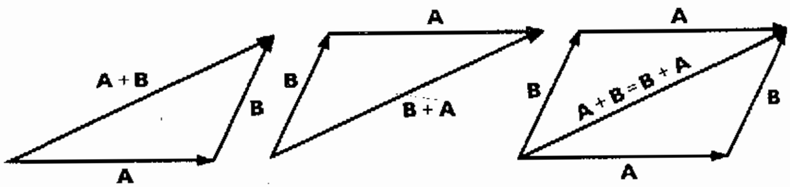


An equivalent way to construct $\mathbf{A} - \mathbf{B}$ is to place the *head* of **B** at the *head* of **A**. Then $\mathbf{A} - \mathbf{B}$ extends from the *tail* of **A** to the *tail* of **B**, as shown in the right hand drawing above.

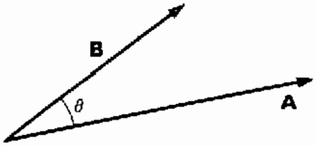
It is not difficult to prove the following laws. We give a geometrical proof of the commutative law; try to cook up your own proofs of the others.

$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$	Commutative law
$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$ $c(d\mathbf{A}) = (cd)\mathbf{A}$	Associative law
$(c + d)\mathbf{A} = c\mathbf{A} + d\mathbf{A}$ $c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$	Distributive law

Proof of the Commutative law of vector addition



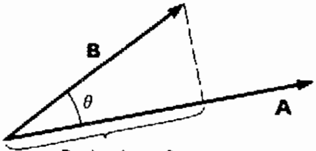
Although there is no great mystery to addition, subtraction, and multiplication of a vector by a scalar, the result of "multiplying" one vector by another is somewhat less apparent. Does multiplication yield a vector, a scalar, or some other quantity? The choice is up to us, and we shall define two types of products which are useful in our applications to physics.



Scalar Product ("Dot" Product) The first type of product is called the *scalar* product, since it represents a way of combining two vectors to form a scalar. The scalar product of **A** and **B** is denoted by $\mathbf{A} \cdot \mathbf{B}$ and is often called the dot product. $\mathbf{A} \cdot \mathbf{B}$ is defined by

$$\mathbf{A} \cdot \mathbf{B} \equiv |\mathbf{A}|\,|\mathbf{B}| \cos\theta.$$

Here θ is the angle between **A** and **B** when they are drawn tail to tail.



Projection of **B** on **A**

Since $|\mathbf{B}| \cos\theta$ is the projection of **B** along the direction of **A**, $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}| \times$ (projection of **B** on **A**).

Similarly,

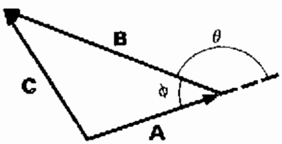
$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{B}| \times (\text{projection of } \mathbf{A} \text{ on } \mathbf{B}).$$

If $\mathbf{A} \cdot \mathbf{B} = 0$, then $|\mathbf{A}| = 0$ or $|\mathbf{B}| = 0$, or **A** is perpendicular to **B** (that is, $\cos\theta = 0$). Scalar multiplication is unusual in that the dot product of two nonzero vectors can be 0.

Note that $\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2$.

By way of demonstrating the usefulness of the dot product, here is an almost trivial proof of the law of cosines.

Example 1.1 **Law of Cosines**



$$\mathbf{C} = \mathbf{A} + \mathbf{B}$$

$$\mathbf{C} \cdot \mathbf{C} = (\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} + \mathbf{B})$$

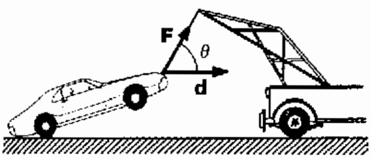
$$|\mathbf{C}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2 + 2|\mathbf{A}|\,|\mathbf{B}| \cos\theta$$

This result is generally expressed in terms of the angle ϕ:

$$C^2 = A^2 + B^2 - 2AB\cos\phi.$$

(We have used $\cos\theta = \cos(\pi - \phi) = -\cos\phi$.)

Example 1.2 **Work and the Dot Product**



The dot product finds its most important application in the discussion of work and energy in Chap. 4. As you may already know, the work W done by a force F on an object is the displacement d of the object times the component of F along the direction of d. If the force is applied at an angle θ to the displacement,

$$W = (F\cos\theta)d.$$

Granting for the time being that force and displacement are vectors,

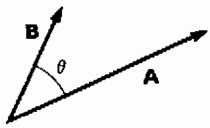
$$W = \mathbf{F} \cdot \mathbf{d}.$$

Vector Product ("Cross" Product) The second type of product we need is the *vector* product. In this case, two vectors **A** and **B** are combined to form a third vector **C**. The symbol for vector product is a cross:

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}.$$

An alternative name is the *cross product.*

The vector product is more complicated than the scalar product because we have to specify both the magnitude and direction of **A** × **B**. The magnitude is defined as follows: if

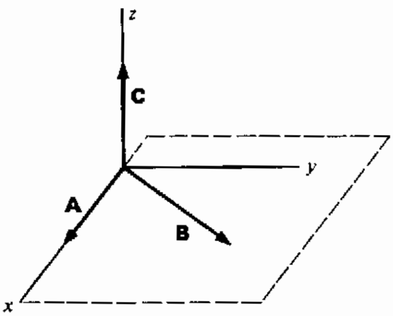


$$\mathbf{C} = \mathbf{A} \times \mathbf{B},$$

then

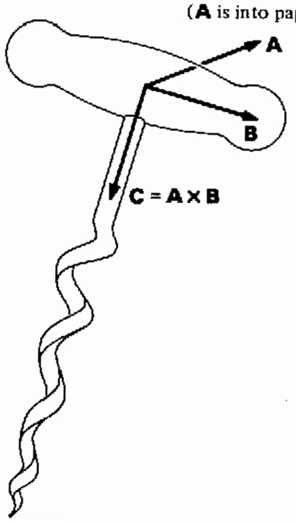
$$|\mathbf{C}| = |\mathbf{A}|\,|\mathbf{B}| \sin\theta,$$

where θ is the angle between **A** and **B** when they are drawn tail to tail. (To eliminate ambiguity, θ is always taken as the angle smaller than π.) Note that the vector product is zero when $\theta = 0$ or π, even if $|\mathbf{A}|$ and $|\mathbf{B}|$ are not zero.



When we draw **A** and **B** tail to tail, they determine a plane. We define the direction of **C** to be perpendicular to the plane of **A** and **B**. **A**, **B**, and **C** form what is called a *right hand triple.* Imagine a right hand coordinate system with **A** and **B** in the xy plane as shown in the sketch. **A** lies on the x axis and **B** lies toward the y axis. If **A**, **B**, and **C** form a right hand triple, then **C** lies on the z axis. We shall always use right hand coordinate systems such as the one shown at left. Here is another way to determine the direction of the cross product. Think of a right hand screw with the axis perpendicular to **A** and **B**. Rotate it in the direction which swings **A** into **B**. **C** lies in the direction the screw advances. (Warning: Be sure not to use a left hand screw. Fortunately, they are rare. Hot water faucets are among the chief offenders; your honest everyday wood screw is right handed.)

(**A** is into paper)



A result of our definition of the cross product is that

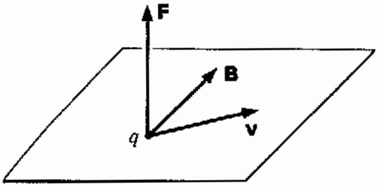
$$\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B}.$$

Here we have a case in which the order of multiplication is important. The vector product is *not* commutative. (In fact, since reversing the order reverses the sign, it is anticommutative.) We see that

$$\mathbf{A} \times \mathbf{A} = 0$$

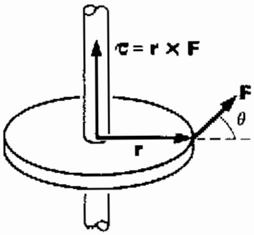
for any vector **A**.

Example 1.3 Examples of the Vector Product in Physics



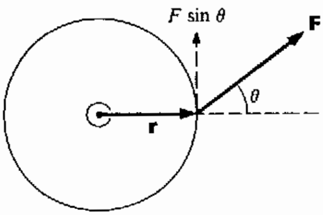
The vector product has a multitude of applications in physics. For instance, if you have learned about the interaction of a charged particle with a magnetic field, you know that the force is proportional to the charge q, the magnetic field B, and the velocity of the particle v. The force varies as the sine of the angle between v and B, and is perpendicular to the plane formed by v and B, in the direction indicated. A simpler way to give all these rules is

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}.$$



Another application is the definition of torque. We shall develop this idea later. For now we simply mention in passing that the torque $\boldsymbol{\tau}$ is defined by

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F},$$

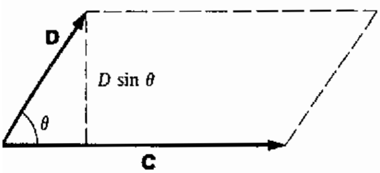


Top view

where $\mathbf{r}$ is a vector from the axis about which the torque is evaluated to the point of application of the force $\mathbf{F}$. This definition is consistent with the familiar idea that torque is a measure of the ability of an applied force to produce a twist. Note that a large force directed parallel to $\mathbf{r}$ produces no twist; it merely pulls. Only $F \sin\theta$, the component of force perpendicular to $\mathbf{r}$, produces a torque. The torque increases as the lever arm gets larger. As you will see in Chap. 6, it is extremely useful to associate a direction with torque. The natural direction is along the axis of rotation which the torque tends to produce. All these ideas are summarized in a nutshell by the simple equation $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$.

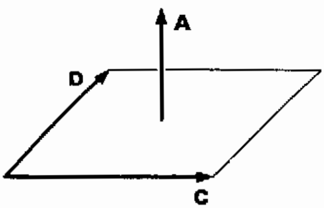
Example 1.4 Area as a Vector

We can use the cross product to describe an area. Usually one thinks of area in terms of magnitude only. However, many applications in physics require that we also specify the orientation of the area. For example, if we wish to calculate the rate at which water in a stream flows through a wire loop of given area, it obviously makes a difference whether the plane of the loop is perpendicular or parallel to the flow. (In the latter case the flow through the loop is zero.) Here is how the vector product accomplishes this:



Consider the area of a quadrilateral formed by two vectors, $\mathbf{C}$ and $\mathbf{D}$. The area of the parallelogram A is given by

$$\begin{aligned} A &= \text{base} \times \text{height} \\ &= CD \sin\theta \\ &= |\mathbf{C} \times \mathbf{D}|. \end{aligned}$$

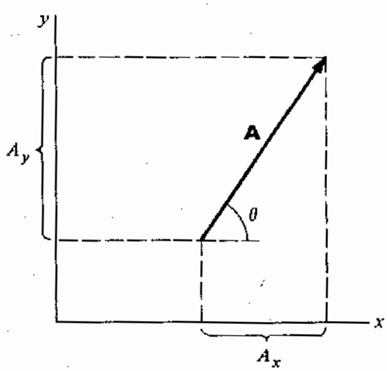


If we think of A as a vector, we have

$$\mathbf{A} = \mathbf{C} \times \mathbf{D}.$$

We have already shown that the magnitude of A is the area of the parallelogram, and the vector product defines the convention for assigning a direction to the area. The direction is defined to be perpendicular to the plane of the area; that is, the direction is parallel to a *normal* to the surface. The sign of the direction is to some extent arbitrary; we could just as well have defined the area by **A** = **D** × **C**. However, once the sign is chosen, it is unique.

1.3 Components of a Vector



The fact that we have discussed vectors without introducing a particular coordinate system shows why vectors are so useful; vector operations are defined without reference to coordinate systems. However, eventually we have to translate our results from the abstract to the concrete, and at this point we have to choose a coordinate system in which to work.

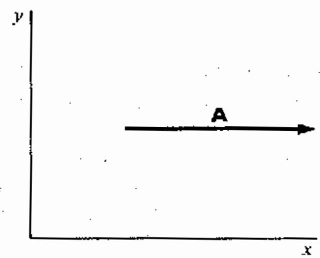
For simplicity, let us restrict ourselves to a two-dimensional system, the familiar xy plane. The diagram shows a vector **A** in the xy plane. The projections of **A** along the two coordinate axes are called the components of **A**. The components of **A** along the x and y axes are, respectively, A_x and A_y. The magnitude of **A** is $|\mathbf{A}| = (A_x^2 + A_y^2)^{\frac{1}{2}}$, and the direction of **A** is such that it makes an angle $\theta = \arctan(A_y/A_x)$ with the x axis.

Since the components of a vector define it, we can specify a vector entirely by its components. Thus

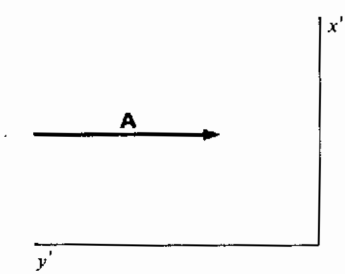
$$\mathbf{A} = (A_x, A_y)$$

or, more generally, in three dimensions,

$$\mathbf{A} = (A_x, A_y, A_z).$$



Prove for yourself that $|\mathbf{A}| = (A_x^2 + A_y^2 + A_z^2)^{\frac{1}{2}}$. The vector **A** has a meaning independent of any coordinate system. However, the components of **A** depend on the coordinate system being used. To illustrate this, here is a vector **A** drawn in two different coordinate systems. In the first case,



$$\mathbf{A} = (A, 0) \qquad (x,y \text{ system}),$$

while in the second

$$\mathbf{A} = (0, -A) \qquad (x',y' \text{ system}).$$

Unless noted otherwise, we shall restrict ourselves to a single coordinate system, so that if

$$\mathbf{A} = \mathbf{B},$$

then

$$A_x = B_x \qquad A_y = B_y \qquad A_z = B_z.$$

The single vector equation $\mathbf{A} = \mathbf{B}$ symbolically represents three scalar equations.

All vector operations can be written as equations for components. For instance, multiplication by a scalar gives

$$c\mathbf{A} = (cA_x, cA_y).$$

The law for vector addition is

$$\mathbf{A} + \mathbf{B} = (A_x + B_x,\ A_y + B_y,\ A_z + B_z).$$

By writing **A** and **B** as the sums of vectors along each of the coordinate axes, you can verify that

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z.$$

We shall defer evaluating the cross product until the next section.

Example 1.5 Vector Algebra

Let

$$\mathbf{A} = (3,5,-7)$$
$$\mathbf{B} = (2,7,1).$$

Find $\mathbf{A} + \mathbf{B}$, $\mathbf{A} - \mathbf{B}$, $|\mathbf{A}|$, $|\mathbf{B}|$, $\mathbf{A} \cdot \mathbf{B}$, and the cosine of the angle between **A** and **B**.

$$\begin{aligned}
\mathbf{A} + \mathbf{B} &= (3 + 2,\ 5 + 7,\ -7 + 1) \\
&= (5,12,-6) \\
\mathbf{A} - \mathbf{B} &= (3 - 2,\ 5 - 7,\ -7 - 1) \\
&= (1,-2,-8) \\
|\mathbf{A}| &= (3^2 + 5^2 + 7^2)^{\frac{1}{2}} \\
&= \sqrt{83} \\
&= 9.11 \\
|\mathbf{B}| &= (2^2 + 7^2 + 1^2)^{\frac{1}{2}} \\
&= \sqrt{54} \\
&= 7.35 \\
\mathbf{A} \cdot \mathbf{B} &= 3 \times 2 + 5 \times 7 - 7 \times 1 \\
&= 34
\end{aligned}$$

$$\cos(\mathbf{A},\mathbf{B}) = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}|\ |\mathbf{B}|} = \frac{34}{(9.11)(7.35)} = 0.507$$

Example 1.6 Construction of a Perpendicular Vector

Find a unit vector in the xy plane which is perpendicular to **A** = (3,5,1)

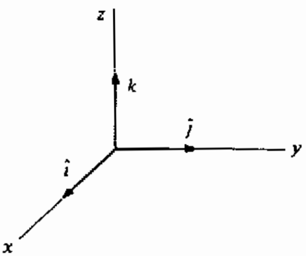
We denote the vector by $\mathbf{B} = (B_x, B_y, B_z)$. Since **B** is in the xy pla $B_z = 0$. For **B** to be perpendicular to **A**, we have $\mathbf{A} \cdot \mathbf{B} = 0$.

$$\mathbf{A} \cdot \mathbf{B} = 3B_x + 5B_y$$
$$= 0$$

Hence $B_y = -\frac{3}{5}B_x$. However, **B** is a unit vector, which means th $B_x^2 + B_y^2 = 1$. Combining these gives $B_x^2 + \frac{9}{25}B_x^2 = 1$, or B_x $\sqrt{\frac{25}{34}} = \pm 0.857$ and $B_y = -\frac{3}{5}B_x = \mp 0.514$.

The ambiguity in sign of B_x and B_y indicates that **B** can point alon line perpendicular to **A** in either of two directions.

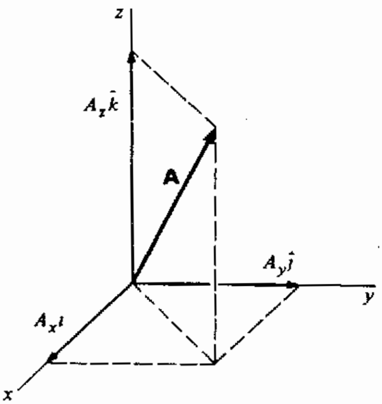
1.4 Base Vectors



Base vectors are a set of orthogonal (perpendicular) unit vecto one for each dimension. For example, if we are dealing with th familiar cartesian coordinate system of three dimensions, the ba vectors lie along the x, y, and z axes. The x unit vector is denot by $\hat{\mathbf{i}}$, the y unit vector by $\hat{\mathbf{j}}$, and the z unit vector by $\hat{\mathbf{k}}$.

The base vectors have the following properties, as you ca readily verify:

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$$
$$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{i}} = 0$$
$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$$
$$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$$
$$\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}.$$



We can write any vector in terms of the base vectors.

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$$

The sketch illustrates these two representations of a vector.

To find the component of a vector in any direction, take the do product with a unit vector in that direction. For instance,

$$A_z = \mathbf{A} \cdot \hat{\mathbf{k}}.$$

It is easy to evaluate the vector product $\mathbf{A} \times \mathbf{B}$ with the aid o the base vectors.

$$\mathbf{A} \times \mathbf{B} = (A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}) \times (B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}})$$

Consider the first term:

$$A_x\hat{\imath} \times \mathbf{B} = A_xB_x(\hat{\imath} \times \hat{\imath}) + A_xB_y(\hat{\imath} \times \hat{\jmath}) + A_xB_z(\hat{\imath} \times \hat{\mathbf{k}}).$$

(We have assumed the associative law here.) Since $\hat{\imath} \times \hat{\imath} = 0$, $\hat{\imath} \times \hat{\jmath} = \hat{\mathbf{k}}$, and $\hat{\imath} \times \hat{\mathbf{k}} = -\hat{\jmath}$, we find

$$A_x\hat{\imath} \times \mathbf{B} = A_x(B_y\hat{\mathbf{k}} - B_z\hat{\jmath}).$$

The same argument applied to the y and z components gives

$$A_y\hat{\jmath} \times \mathbf{B} = A_y(B_z\hat{\imath} - B_x\hat{\mathbf{k}})$$
$$A_z\hat{\mathbf{k}} \times \mathbf{B} = A_z(B_x\hat{\jmath} - B_y\hat{\imath}).$$

A quick way to derive these relations is to work out the first and then to obtain the others by cyclically permuting x, y, z, and $\hat{\imath}$, $\hat{\jmath}$, $\hat{\mathbf{k}}$ (that is, $x \to y$, $y \to z$, $z \to x$, and $\hat{\imath} \to \hat{\jmath}$, $\hat{\jmath} \to \hat{\mathbf{k}}$, $\hat{\mathbf{k}} \to \hat{\imath}$.) A simple way to remember the result is to use the following device: write the base vectors and the components of **A** and **B** as three rows of a determinant,[1] like this

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$
$$= \hat{\imath}(A_yB_z - A_zB_y) - \hat{\jmath}(A_xB_z - A_zB_x) + \hat{\mathbf{k}}(A_xB_y - A_yB_x).$$

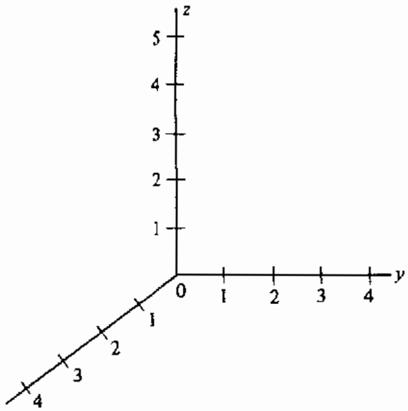
For instance, if $\mathbf{A} = \hat{\imath} + 3\hat{\jmath} - \hat{\mathbf{k}}$ and $\mathbf{B} = 4\hat{\imath} + \hat{\jmath} + 3\hat{\mathbf{k}}$, then

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 1 & 3 & -1 \\ 4 & 1 & 3 \end{vmatrix}$$
$$= 10\hat{\imath} - 7\hat{\jmath} - 11\hat{\mathbf{k}}.$$

1.5 Displacement and the Position Vector

So far we have discussed only abstract vectors. However, the reason for introducing vectors here is concrete—they are just right for describing kinematical laws, the laws governing the geometrical properties of motion, which we need to begin our discussion of mechanics. Our first application of vectors will be to the description of position and motion in familiar three dimensional space. Although our first application of vectors is to the motion of a point in space, don't conclude that this is the only

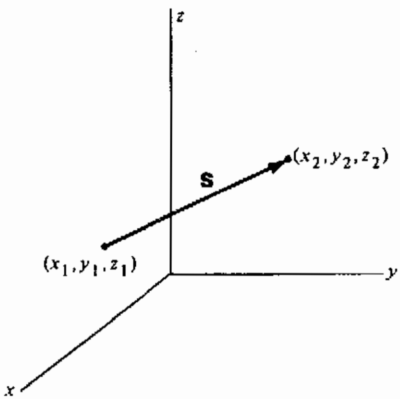
[1] If you are unfamiliar with simple determinants, most of the books listed at the end of the chapter discuss determinants.



application, or even an unusually important one. Many physical quantities besides displacements are vectors. Among these are velocity, force, momentum, and gravitational and electric fields.

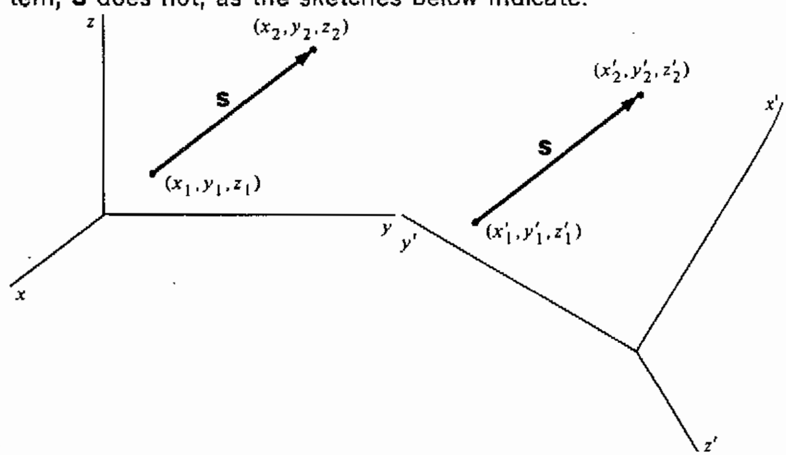
To locate the position of a point in space, we start by setting up a coordinate system. For convenience we choose a three dimensional cartesian system with axes x, y, and z, as shown.

In order to measure position, the axes must be marked off in some convenient unit of length—meters, for instance.

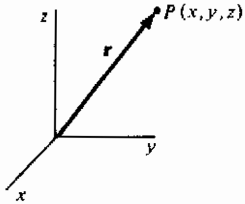


The position of the point of interest is given by listing the values of its three coordinates, x_1, y_1, z_1. These numbers do *not* represent the components of a vector according to our previous discussion. (They specify a position, not a magnitude and direction.) However, if we move the point to some new position, x_2, y_2, z_2, then the *displacement* defines a vector **S** with coordinates $S_x = x_2 - x_1$, $S_y = y_2 - y_1$, $S_z = z_2 - z_1$.

S is a vector from the initial position to the final position—it defines the displacement of a point of interest. Note, however, that **S** contains no information about the initial and final positions separately—only about the *relative* position of each. Thus, $S_z = z_2 - z_1$ depends on the *difference* between the final and initial values of the z coordinates; it does not specify z_2 or z_1 separately. **S** is a true vector; although the values of the coordinates of the initial and final points depend on the coordinate system, **S** does not, as the sketches below indicate.

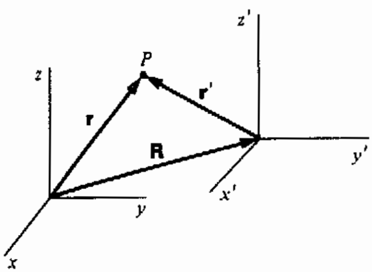


One way in which our displacement vector differs from a mathematician's vector is that his vectors are usually pure quantities, with components given by absolute numbers, whereas **S** has the physical dimension of length associated with it. We will use the convention that the magnitude of a vector has dimensions so that a unit vector is dimensionless. Thus, a displacement of 8 m (8 meters) in the x direction is $\mathbf{S} = (8\text{ m}, 0, 0)$. $|\mathbf{S}| = 8$ m, and $\hat{\mathbf{S}} = \mathbf{S}/|\mathbf{S}| = \hat{\mathbf{i}}$.



Although vectors define displacements rather than positions, it is in fact possible to describe the position of a point with respect to the origin of a given coordinate system by a special vector, known as the *position vector,* which extends from the origin to the point of interest. We shall use the symbol $\mathbf{r}$ to denote the position vector. The position of an arbitrary point P at (x,y,z) is written as

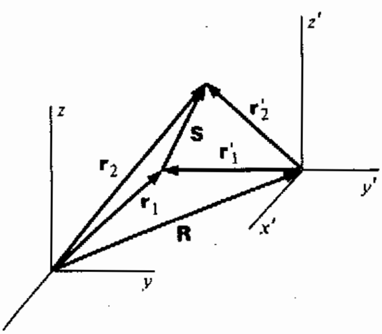
$$\mathbf{r} = (x,y,z) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}.$$



Unlike ordinary vectors, $\mathbf{r}$ depends on the coordinate system. The sketch to the left shows position vectors $\mathbf{r}$ and $\mathbf{r}'$ indicating the position of the same point in space but drawn in different coordinate systems. If $\mathbf{R}$ is the vector from the origin of the unprimed coordinate system to the origin of the primed coordinate system, we have

$$\mathbf{r}' = \mathbf{r} - \mathbf{R}.$$

In contrast, a true vector, such as a displacement $\mathbf{S}$, is independent of coordinate system. As the bottom sketch indicates,



$$\begin{aligned}\mathbf{S} &= \mathbf{r}_2 - \mathbf{r}_1 \\ &= (\mathbf{r}_2' + \mathbf{R}) - (\mathbf{r}_1' + \mathbf{R}) \\ &= \mathbf{r}_2' - \mathbf{r}_1'.\end{aligned}$$

1.6 Velocity and Acceleration

Motion in One Dimension

Before applying vectors to velocity and acceleration in three dimensions, it may be helpful to review briefly the case of one dimension, motion along a straight line.

Let x be the value of the coordinate of a particle moving along a line. x is measured in some convenient unit, such as meters, and we assume that we have a continuous record of position versus time.

The *average velocity* $\bar{v}$ of the point between two times, t_1 and t_2, is defined by

$$\bar{v} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}.$$

(We shall often use a bar to indicate an average of a quantity.)

The *instantaneous velocity* v is the limit of the average velocity as the time interval approaches zero.

$$v = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}.$$

The limit we have introduced in defining v is precisely that involved in the definition of a derivative. In fact, we have[1]

$$v = \frac{dx}{dt}.$$

In a similar fashion, the *instantaneous acceleration* is

$$a = \lim_{\Delta t \to 0} \frac{v(t + \Delta t) - v(t)}{\Delta t}$$
$$= \frac{dv}{dt}.$$

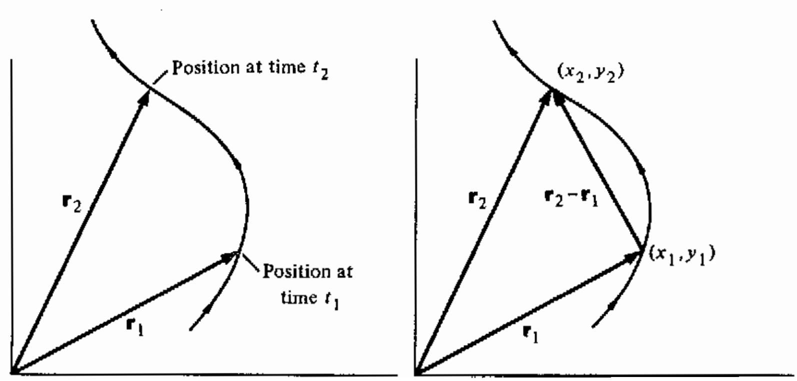
The concept of speed is sometimes useful. Speed s is simply the magnitude of the velocity: $s = |\mathbf{v}|$.

Motion in Several Dimensions

Our task now is to extend the ideas of velocity and acceleration to several dimensions. Consider a particle moving in a plane. As time goes on, the particle traces out a path, and we suppose that we know the particle's coordinates as a function of time. The instantaneous position of the particle at some time t_1 is

$$\mathbf{r}(t_1) = [x(t_1), y(t_1)] \qquad \text{or} \qquad \mathbf{r}_1 = (x_1, y_1),$$

[1] Physicists generally use the Leibnitz notation dx/dt, since this is a handy form for using differentials (see Note 1.1). Starting in Sec. 1.9 we shall use Newton's notation $\dot{x}$, but only to denote derivatives with respect to time.

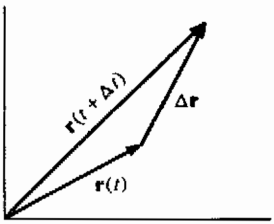


where x_1 is the value of x at $t = t_1$, and so forth. At time t_2 the position is

$$\mathbf{r}_2 = (x_2, y_2).$$

The displacement of the particle between times t_1 and t_2 is

$$\mathbf{r}_2 - \mathbf{r}_1 = (x_2 - x_1,\ y_2 - y_1).$$

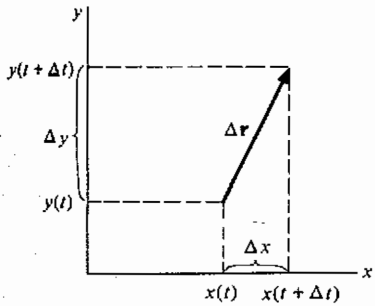


We can generalize our example by considering the position at some time t, and at some later time $t + \Delta t$.† The displacement of the particle between these times is

$$\Delta\mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t).$$

This vector equation is equivalent to the two scalar equations

$$\Delta x = x(t + \Delta t) - x(t)$$
$$\Delta y = y(t + \Delta t) - y(t).$$



The *velocity* $\mathbf{v}$ of the particle as it moves along the path is defined to be

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt},$$

which is equivalent to the two scalar equations

$$v_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$
$$v_y = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dt}.$$

Extension of the argument to three dimensions is trivial. The third component of velocity is

$$v_z = \lim_{\Delta t \to 0} \frac{z(t + \Delta t) - z(t)}{\Delta t} = \frac{dz}{dt}.$$

Our definition of velocity as a vector is a straightforward generalization of the familiar concept of motion in a straight line. Vector notation allows us to describe motion in three dimensions with a single equation, a great economy compared with the three equations we would need otherwise. The equation $\mathbf{v} = d\mathbf{r}/dt$ expresses the results we have just found.

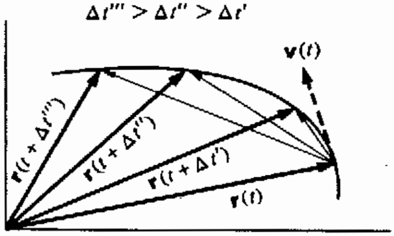
† We will often use the quantity Δ to denote a difference or change, as in the case here of $\Delta\mathbf{r}$ and Δt. However, this implies nothing about the size of the quantity, which may be large or small, as we please.

Alternatively, since $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, we obtain by simple differentiation[1]

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}}$$

as before.

Let the particle undergo a displacement $\Delta\mathbf{r}$ in time Δt. In the limit $\Delta t \to 0$, $\Delta\mathbf{r}$ becomes tangent to the trajectory, as the sketch indicates. However, the relation



$$\Delta\mathbf{r} \approx \frac{d\mathbf{r}}{dt}\Delta t$$
$$= \mathbf{v}\,\Delta t,$$

which becomes exact in the limit $\Delta t \to 0$, shows that $\mathbf{v}$ is parallel to $\Delta\mathbf{r}$; the instantaneous velocity $\mathbf{v}$ of a particle is everywhere tangent to the trajectory.

Example 1.7 Finding v from r

The position of a particle is given by

$$\mathbf{r} = A(e^{\alpha t}\hat{\mathbf{i}} + e^{-\alpha t}\hat{\mathbf{j}}),$$

where α is a constant. Find the velocity, and sketch the trajectory.

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$
$$= A(\alpha e^{\alpha t}\hat{\mathbf{i}} - \alpha e^{-\alpha t}\hat{\mathbf{j}})$$

or

$$v_x = A\alpha e^{\alpha t}$$
$$v_y = -A\alpha e^{-\alpha t}.$$

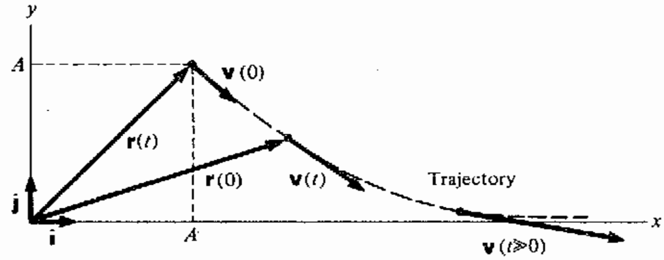
The magnitude of $\mathbf{v}$ is

$$v = (v_x^2 + v_y^2)^{\frac{1}{2}}$$
$$= A\alpha(e^{2\alpha t} + e^{-2\alpha t})^{\frac{1}{2}}.$$

In sketching the motion of a point, it is usually helpful to look at limiting cases. At $t = 0$, we have

$$\mathbf{r}(0) = A(\hat{\mathbf{i}} + \hat{\mathbf{j}})$$
$$v(0) = \alpha A(\hat{\mathbf{i}} - \hat{\mathbf{j}}).$$

[1] Caution: We can neglect the cartesian unit vectors when we differentiate, since their directions are fixed. Later we shall encounter unit vectors which can change direction, and then differentiation is more elaborate.



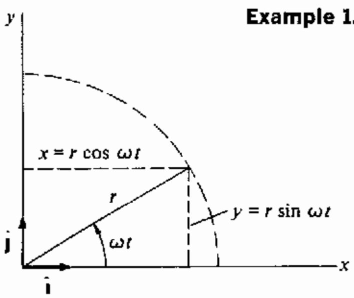
As $t \to \infty$, $e^{\alpha t} \to \infty$ and $e^{-\alpha t} \to 0$. In this limit $\mathbf{r} \to Ae^{\alpha t}\hat{\mathbf{i}}$, which is a vector along the x axis, and $\mathbf{v} \to \alpha Ae^{\alpha t}\hat{\mathbf{i}}$; the speed increases without limit.

Similarly, the acceleration **a** is defined by

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{dv_x}{dt}\hat{\mathbf{i}} + \frac{dv_y}{dt}\hat{\mathbf{j}} + \frac{dv_z}{dt}\hat{\mathbf{k}}$$

$$= \frac{d^2\mathbf{r}}{dt^2}.$$

We could continue to form new vectors by taking higher derivatives of **r**, but we shall see in our study of dynamics that **r**, **v**, and **a** are of chief interest.



Example 1.8 **Uniform Circular Motion**

Circular motion plays an important role in physics. Here we look at the simplest and most important case—*uniform* circular motion, which is circular motion at constant speed.

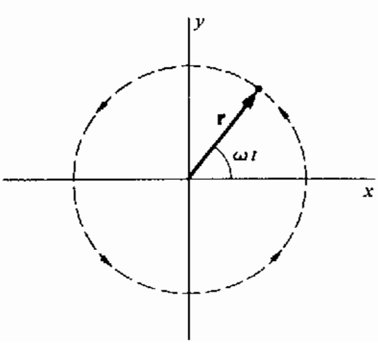
Consider a particle moving in the xy plane according to $\mathbf{r} = r(\cos \omega t\hat{\mathbf{i}} + \sin \omega t\hat{\mathbf{j}})$, where r and ω are constants. Find the trajectory, the velocity, and the acceleration.

$$|\mathbf{r}| = [r^2 \cos^2 \omega t + r^2 \sin^2 \omega t]^{\frac{1}{2}}$$

Using the familiar identity $\sin^2 \theta + \cos^2 \theta = 1$,

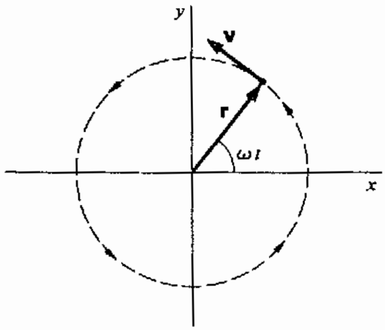
$$|\mathbf{r}| = [r^2(\cos^2 \omega t + \sin^2 \omega t)]^{\frac{1}{2}}$$

$$= r = \text{constant}.$$



The trajectory is a circle.

The particle moves counterclockwise around the circle, starting from $(r,0)$ at $t = 0$. It traverses the circle in a time T such that $\omega T = 2\pi$. ω is called the *angular velocity* of the motion and is measured in radians



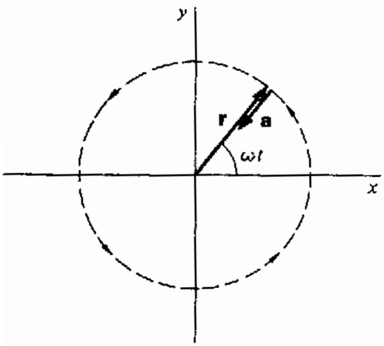
per second. T, the time required to execute one complete cycle, is called the *period.*

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$
$$= r\omega(-\sin \omega t\hat{\mathbf{i}} + \cos \omega t\hat{\mathbf{j}})$$

We can show that **v** is tangent to the trajectory by calculating $\mathbf{v} \cdot \mathbf{r}$:

$$\mathbf{v} \cdot \mathbf{r} = r^2\omega(-\sin \omega t \cos \omega t + \cos \omega t \sin \omega t)$$
$$= 0.$$

Since **v** is perpendicular to **r**, it is tangent to the circle as we expect. Incidentally, it is easy to show that $|\mathbf{v}| = r\omega =$ constant.



$$\mathbf{a} = \frac{d\mathbf{v}}{dt}$$
$$= r\omega^2[-\cos \omega t\hat{\mathbf{i}} - \sin \omega t\hat{\mathbf{j}}]$$
$$= -\omega^2\mathbf{r}$$

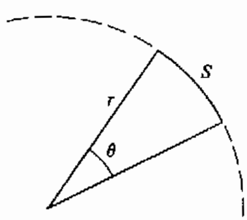
The acceleration is directed radially inward, and is known as the *centripetal acceleration.* We shall have more to say about it shortly.

A Word about Dimension and Units

Physicists call the fundamental physical units in which a quantity is measured the *dimension* of the quantity. For example, the dimension of velocity is distance/time and the dimension of acceleration is velocity/time or (distance/time)/time = distance/time². As we shall discuss in Chap. 2, mass, distance, and time are the fundamental physical units used in mechanics.

To introduce a system of units, we specify the standards of measurement for mass, distance, and time. Ordinarily we measure distance in meters and time in seconds. The units of velocity are then meters per second (m/s) and the units of acceleration are meters per second² (m/s²).

The natural unit for measuring angle is the *radian* (rad). The angle θ in radians is S/r, where S is the arc subtended by θ in a circle of radius **r**:



$$\theta = \frac{S}{r}.$$

2π rad = 360°. We shall always use the radian as the unit of angle, unless otherwise stated. For example, in $\sin \omega t$, ωt is in radians. ω therefore has the dimensions 1/time and the units