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2D Trajectories and Some Math

I. HW#3, Problem 2

A ball is dropped from a height H. The wind is blowing horizontally and imparts a constant acceleration a_0 on the ball.

a) Show that the path of the ball is a straight line and find the values of R and θ in the diagram:
(draw: $+\hat{x}$ pointing right, $+\hat{y}$ pointing up.)

b) How long does it take for the ball to reach the ground?

c) With what speed does the ball hit the ground?

Solution:

As a precursor to the problem, set up a coordinate system and solve for the position vector $\vec{x}(t)$ of the ball as a function of time. Choose the time parameter such that the ball is dropped at $t = 0$, and the ball is sitting at rest for $t < 0$.

$$\vec{a} = a_0\hat{x} - g\hat{y} \implies \vec{v}(t) = \int_0^t dt'\, \vec{a} = a_0 t\, \hat{x} - gt\, \hat{y} + \vec{v}_0$$

$$\vec{x}(t) = \int_0^t dt'\, \vec{v}(t') = \frac{1}{2}a_0 t^2\, \hat{x} - \frac{1}{2}gt^2\, \hat{y} + \vec{v}_0 t + \vec{x}_0$$

Mathematically, $\vec{v}_0$ and $\vec{x}_0$ are simply integration constants. Physically, $\vec{v}_0$ is the velocity vector of the ball at time $t = 0$, and $\vec{x}_0$ is the position of the ball at time $t = 0$. Since the ball is dropped from rest, $\vec{v}_0 = \vec{0}$. The ball starts at $x = 0$ and $y = H$, so $\vec{x}_0 = H\hat{y}$. The position of the ball is

$$\vec{x}(t) = \frac{1}{2}a_0 t^2\, \hat{x} + (H - \frac{1}{2}gt^2)\hat{y}\ .$$

The x component of the position is $x(t) \equiv \hat{x} \cdot \vec{x}(t)$ (bad notation, I know), so $x(t) = \frac{1}{2}a_0 t^2$. The y component of the position is $y(t) \equiv \hat{y} \cdot \vec{x}(t) = H - \frac{1}{2}gt^2$.

Now let's do part (b). Define the time T to hit the ground by the condition $y(T) = 0$. Since $y(t) = H - \frac{1}{2}gt^2$, the condition $y(T) = 0$ implies $H - \frac{1}{2}gT^2 = 0$. Therefore the time for the ball to hit the ground is

$$T = \sqrt{\frac{2H}{g}}\ .$$

A useful way to check for errors is to make sure the dimensions are the same on the left and on the right. The thing on the left is a time, so it has dimensions of time. If the thing on the right does not have dimensions of time, then we have made a mistake.[1] Let square brackets [...] denote the dimension of something. For example, we write "$[T]$ = time". Since $[g]$ = length/time^2 and $[H]$ = length, we have $[H/g] = (\text{length})\frac{\text{time}^2}{\text{length}} = \text{time}^2$, so $[\sqrt{H/g}] = $ time, which is correct.

Now part (a). The trajectory is a straight line because both $x(t)$ and $y(t)$ change with time as t^2. Explicitly, we can rearrange $x = \frac{1}{2}a_0 t^2$ to get $t = \sqrt{2x/a_0}$, then plug this into $y(t) = H - \frac{1}{2}gt^2$ to get $y(x)$:

$$y(x) \equiv y(t = \sqrt{2x/a_0}\,) = H - \frac{1}{2}g(2x/a_0) = H - \frac{g}{a_0}x\ .$$

This is the equation of a straight line with slope $-g/a_0$.

As defined in the diagram, R is the distance traveled from time $t = 0$ to time $t = T$ when the ball hits the ground, so $R = x(T) = \frac{1}{2}a_0 T^2$. Since $T = \sqrt{2H/g}$, we have

$$R = \frac{1}{2}a_0\left(\sqrt{\frac{2H}{g}}\right)^2 = \frac{a_0}{g}H\ .$$

Again, check dimensions. Since a_0 and g both have dimensions of acceleration, their ratio a_0/g is dimensionless, so $[R] = [H]$ = length, which is correct. The angle θ is defined by $\tan\theta = H/R = H/(a_0 H/g) = g/a_0$, which is dimensionless. Therefore $\theta = \tan^{-1}(g/a_0)$.

Now part (c). The speed $v(t)$ is the square root of $v(t)^2 \equiv \vec{v}(t)\cdot\vec{v}(t) = (a_0 t\,\hat{x} - gt\,\hat{y})\cdot(a_0 t\,\hat{x} - gt\,\hat{y}) = (a_0 t)^2 + (-gt)^2 = (a_0^2 + g^2)t^2$. Therefore the speed at time t is $v(t) = \sqrt{a_0^2 + g^2}\,t$. Using $T = \sqrt{2H/g}$, we find that the speed of the ball when it hits the ground is

$$v(T) = \sqrt{\left(\frac{a_0^2 + g^2}{g}\right)2H}\ .$$

Please get in the habit of checking the dimensions of every answer you get. This incredibly simple process of dimensional analysis plays a rather shockingly large role in every aspect of modern physics, including biophysics, condensed matter theory, and particle physics. Anyway, $[(a_0^2 + g^2)/g] = [g]$ =length/time^2, and $[H]$ = length, so $\sqrt{(a_0^2 + g^2)\frac{2H}{g}}$ has dimensions of $\sqrt{\text{length}^2/\text{time}^2}$ = length/time, which indeed is the dimensions for speed.

II. Approximations and Gaussian Integrals.

Now let's do something totally unrelated. If you understand conceptually what is about to follow, then you will find at some point in your life that you run across a kinematics problem in which this will be useful. But for now, it's just for fun.

[1] Of course, the converse is not true. Just because the dimensions work out doesn't mean you got the right answer! For example, you can't predict the factor of 2 from dimensional analysis.

The goal is to evaluate the integral

$$I \equiv \int_{-\infty}^{\infty} dq\; e^{-f(q)} \;,$$

where $f(q)$ is a function that looks like
(draw)
but whose explicit form will not be given. The important features are: 1) The function $f(q)$ has a deep minimum at $q = q_*$. 2) Away from $q = q_*$ the function $f(q)$ goes to infinity, so that $\lim_{q\to\pm\infty} e^{-f(q)} = 0$.

Well, since I refuse to give you the function $f(q)$ except for the above features, we can't do the integral. But what we can do is derive an approximate form for the integral. Consider the Taylor series of $f(q)$ about the point $q = q_*$:

$$f(q) = f(q_*) + (q - q_*)f'(q_*) + \frac{1}{2}(q - q_*)^2 f''(q_*) + O(q - q_*)^3 \;.$$

The notation $O(x)$ means "of order x," so $O(q - q_*)^3$ means "of order $(q - q_*)^3$ ". Since $q = q_*$ is a minimum of $f(q)$, its first derivative is zero there: $f(q_*) = 0$. We will now drop all terms of order $(q - q_*)^3$ and higher, and therefore *approximate* the function $f(q)$ as

$$f(q) \approx f(q_*) + \frac{1}{2}(q - q_*)^2 f''(q_*) \;.$$

Now our integral is approximately equal to

$$I \approx J \equiv \int_{-\infty}^{\infty} dq\, e^{-f(q_*) - \frac{1}{2} f''(q_*)(q - q_*)^2} \;.$$

Since $f(q_*)$ is a constant, we can pull the factor $e^{-f(q_*)}$ out of the integral. Since the bounds are $\pm\infty$, we can shift the integration variable q by a constant using the substitution $x \equiv q - q_*$ to get

$$J = e^{-f(q_*)} \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2} f''(q_*) x^2} \;.$$

To the untrained eye[2], this integral looks no better than the one we started with. But in fact it can be computed exactly! If I have time, I will show you that

$$\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}} \quad \text{for} \quad \alpha > 0$$

but the important point is that if you put that integral into a computer program like Mathematica, it will spit out an answer for you. If we write $\alpha = \frac{1}{2} f''(q_*)$, which is indeed positive since q_* is a minimum of $f(q)$, then we have

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2} f''(q_*) x^2} = \sqrt{\frac{2\pi}{f''(q_*)}} \;.$$

[2] namely a freshman just out of high school. :)

Therefore, we have an approximate answer to the original question:

$$\int_{-\infty}^{\infty} dq\, e^{-f(q)} \approx e^{-f(q_*)} \sqrt{\frac{2\pi}{f''(q_*)}} \,.$$

For fun, I will tantalize you with the fact that you are now ready to derive an approximate expression for the factorial function. Recall the definition of the factorial: $n! \equiv n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$. For really large n, say 100, this gets very tedious very quickly. With some help from Wikipedia, you are now ready to derive the result

$$n! \approx \sqrt{2\pi\, n}\, n^n\, e^{-n} \,,$$

which is known as "Stirling's formula" and is a good approximation when n is large.