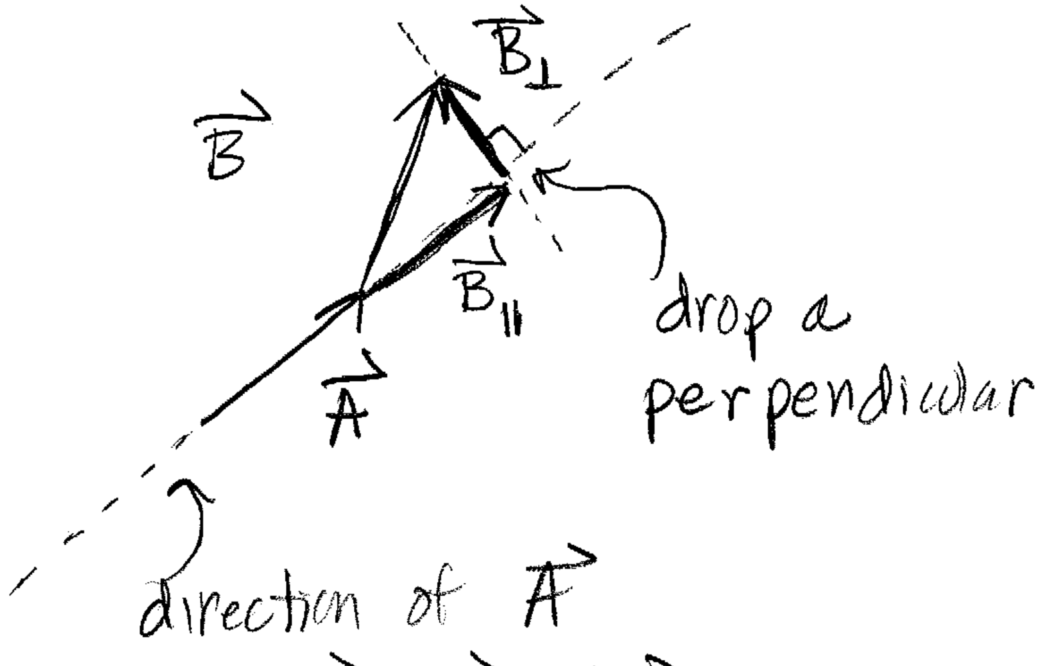
The Geometric Analysis of this Inequality Qualitatively... Heart of the Matter is the <u>Pythagorean</u> <u>Theorem</u>

Useful to <u>Break</u> $\vec{B}$ into a piece <u>Parallel to $\vec{A}$</u> + $\underbrace{\text{Perpendicular to } \vec{A}}$

this does not exist in One dimension

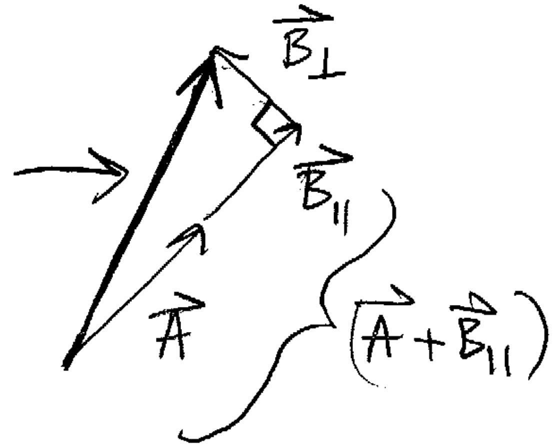


Conceptually, $\vec{B} = \underbrace{\vec{B}_\parallel + \vec{B}_\perp}$

"decomposing $\vec{B}$ into directions $\parallel$ + $\perp$ to $\vec{A}$"

$$\vec{A} + \vec{B} = \underbrace{\vec{A} + \vec{B}_\parallel}_{\text{like 1-d}} + \underbrace{\vec{B}_\perp}_{\text{Pythagorean!}}$$

$$|\vec{A}+\vec{B}| = \underbrace{\sqrt{|\vec{A}+\vec{B}_{\parallel}|^2 + |\vec{B}_{\perp}|^2}}_{\text{Pythagoras}}$$



Pythagoras says: add in quadrature

Adding in quadrature gives a smaller result than regular adding

Regular adding : $1 + 1 = 2$

Quadrature adding : $\sqrt{1^2+1^2} = \sqrt{2} < 2$

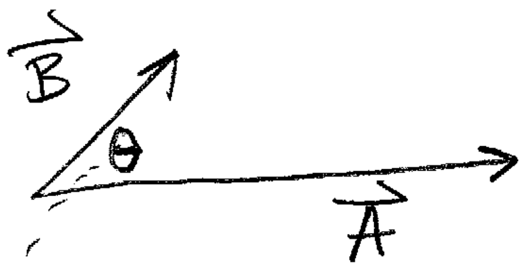
That Pythagoras "said" to add in add in quadrature is the "heart" of why:

$$|\vec{A}+\vec{B}| \leq |\vec{A}| + |\vec{B}|$$

equality only one 1-dimension sufficient.

This situation is better explored/discussed with the dot product or scalar product.

Geometric Definition of Dot Product



put $\vec A + \vec B$ $\boxed{\text{TAIL TO TAIL}}$

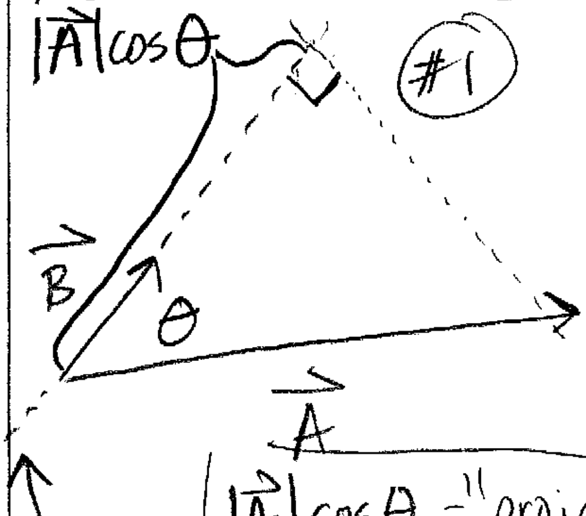
not like addition!

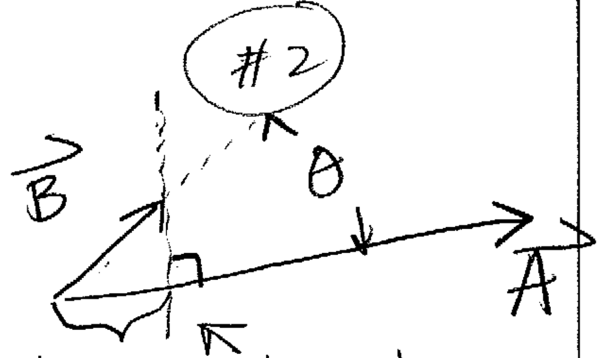
direction of θ ↺ ↻ irrelevant.

$$\vec A \cdot \vec B = \vec B \cdot \vec A \equiv |\vec A||\vec B| \cos\theta$$

$\boxed{\text{definition of dot product}}$ defined Geometric $\cos\theta = \cos(-\theta)$

Two interpretations





$|\vec B|\cos\theta$ this time, $\perp$ to line of A

extend "line of $\vec B$"

$|\vec A|\cos\theta$ = "projection of $\vec A$ on $\vec B$"

$|\vec B|\cos\theta$ = "projection of $\vec B$ on $\vec A$"

$|\vec B| \cdot |\vec A| \cos\theta = \vec A \cdot \vec B$ $\vec A \cdot \vec B = |\vec A||\vec B| \cos\theta$

What does $\vec{A} \cdot \vec{B} < 0$ mean?

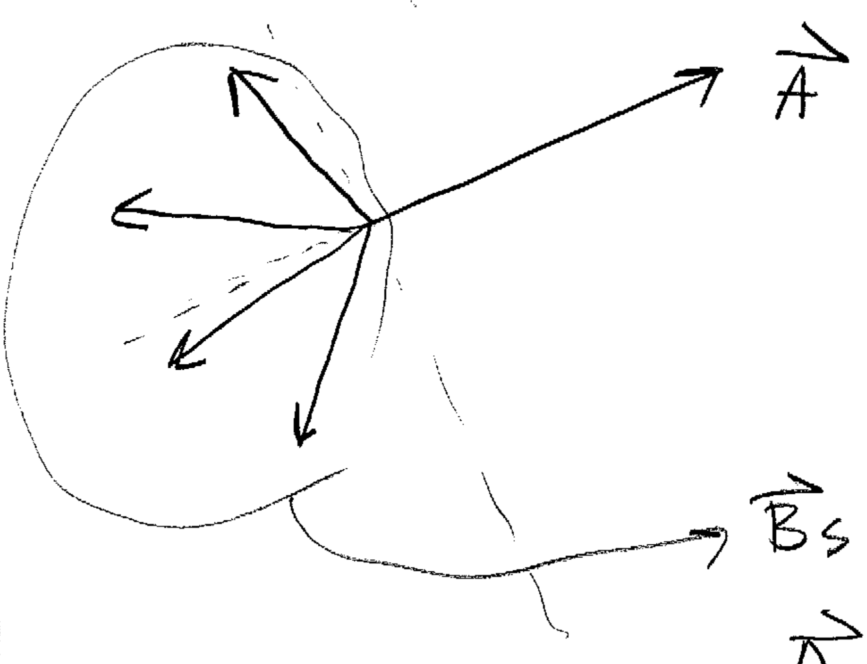
$$|\vec{A}||\vec{B}| \cos\theta < 0$$

$$\cos\theta < 0$$

$$\left.\begin{array}{ll} & \frac{\pi}{2} < \theta < \frac{3}{2}\pi \\ \text{or} & \frac{\pi}{2} < \theta < \pi \\ & -\pi < \theta < -\frac{\pi}{2} \end{array}\right\} \theta \text{ in quadrants II, III.}$$

II	I
III	IV

$\perp$ to $\vec{A}$ at its base



$\vec{B}$s that have $\vec{A} \cdot \vec{B} < 0$

$$\vec{A} \cdot \vec{A} = |\vec{A}| \cdot |\vec{A}| \cdot \cos(0) = |\vec{A}|^2$$

$$|\vec{A} + \vec{B}|^2 = \underbrace{(\vec{A} + \vec{B}) \cdot (\vec{A} + \vec{B})}_{\text{dot product distributes, commutes}}$$

$$|\vec{A}+\vec{B}|^2 = \vec{A}\cdot\vec{A} + \vec{A}\cdot\vec{B} + \vec{B}\cdot\vec{A} + \vec{B}\cdot\vec{B}$$

$$= |\vec{A}|^2 + |\vec{B}|^2 + 2|\vec{A}||\vec{B}|\cos\theta$$

Maximum $\quad \cos\theta = 1 \quad$ (θ real-valued)

$$|\vec{A}+\vec{B}| \leq |\vec{A}|^2 + |\vec{B}|^2 + 2|\vec{A}||\vec{B}|$$

$$\leq (|\vec{A}| + |\vec{B}|)^2$$

Minimum $\quad \cos\theta = -1$

$$|\vec{A}+\vec{B}| \geq |\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}||\vec{B}|$$

$$\geq (|\vec{A}| - |\vec{B}|)^2$$

Triangle Inequality ... equalities

$\cos\theta = \pm 1$

$\vec{A}, \vec{B}$ $\parallel$ or $\uparrow\downarrow$

Cross Product (into the third dimension!)

Dot Product: 2 vectors $\rightarrow$ result is a scalar

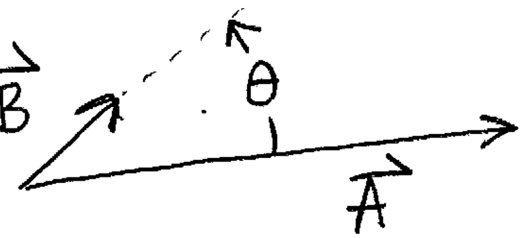
Cross Product 2 vectors $\rightarrow$ result is a vector, in the 3rd dimension!

Order of vectors matters in cross product

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \quad (\text{"anti-commute"})$$

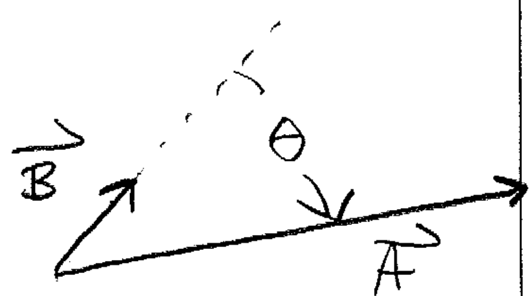
Geometric Cross Product Definition

- Put vectors tail-to-tail
- Orientation of θ, angle between, now matters



$\vec{A} \times \vec{B}$: θ from $\vec{A}$ to $\vec{B}$

this $\theta > 0$



$\vec{B} \times A$: θ from $\vec{B}$ to $\vec{A}$

this $\theta < 0$

$\vec{A} \times \vec{B}$ is a vector

its $\boxed{\text{component}}$ out (+) or in (−) of page

$= |\vec{A}||\vec{B}| \sin\theta > 0.$

"page" $\equiv$ plane containing $\vec{A}$ + $\vec{B}$

$\vec{A} \times \vec{B}$ is out of page.

$|\vec{A}||\vec{B}| \sin\theta$

here $\sin\theta < 0$

$\vec{B} \times \vec{A}$ is in to page

<u>Notation</u> :

IN TO PAGE $\otimes$

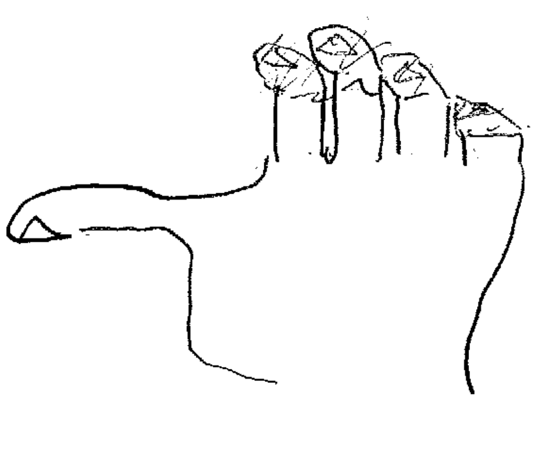
back of arrow

OUT OF PAGE $\odot$

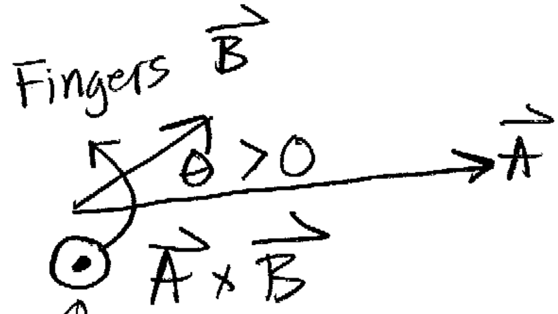
Follows "Right Hand Rule"

① Align your right hand

② Practice Curling Fingers Forward

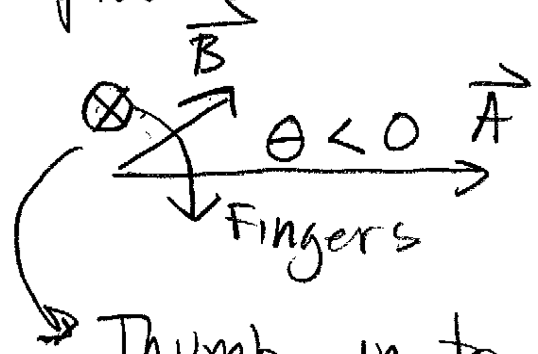


(3) Align Curled Fingers so they point from first vector in cross product to second; Thumb points in direction of cross product



Thumb, out of page, positive

$\sin\theta > 0$



Thumb, in to page, negative

$\sin\theta < 0$

And so: $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

$$\vec{A} \times \vec{A} = \vec{B} \times \vec{B} = 0$$

Base Vectors in 2-d, 3-d

We talked about $\hat{\imath} \rightarrow$ base vector in one dimension (sometimes called $\hat{x}$).

$\hat{\imath} \rightarrow$ no dimensions, $|\hat{\imath}| = 1$