1. All force directions toward center, or, $-\vec{F}$

(a) \[ \vec{F} = -\frac{G(M_1 + M_2)m}{a^2} \]

since shells uniform and all mass at radii < $a$, as if all mass concentrated at origin.

(b) \[ \vec{F} = -\frac{GM_1m}{b^2} \]

(c) \[ \vec{F} = 0 \] no mass inside

3. radius of hole is $-\frac{R}{2}$

use concept of superposition...

"LIKE" full sphere of radius $R$ + $-\frac{mass}{sphere}$ of $\frac{1}{8}$ mass at $\frac{R}{2}$
\[ F = \left( -\frac{GMm}{a^2} + \frac{G\left(\frac{M}{8}\right)m}{(a - R/2)^2} \right) \hat{r} \quad \text{unit away from center} \]

\[ F = -\frac{GMm}{a^2} \left[ 1 - \frac{1}{8(1 - R/2a)^2} \right] \hat{r} \]

\[ r_E = r_M + 25 \text{ km} \]
\[ r_E = 6370 \text{ km} \]

\[ \rho_c = 3490 \text{ km} \]

4

\[ M_{\text{total}} = M_{\text{crust}} + M_{\text{mantle}} + M_{\text{core}} \]
\[ = (0.0394 + 4.01 + 1.93) \cdot 10^{24} \text{ kg} \]
\[ = 5.98 \cdot 10^{24} \text{ kg} \]

\[ g = \frac{GM_{\text{total}}}{R_E^2} = \frac{(6.673 \cdot 10^{-11})(5.98 \cdot 10^{24})}{(6370 \cdot 10^3)^2} \]
\[ g = 9.83 \text{ m/s}^2 \]

(a) \[ M_{\text{inside}} = M_{\text{mantle}} + M_{\text{core}} \]
\[ = (4.01 + 1.93) \cdot 10^{24} \text{ kg} \]
\[ = 5.94 \cdot 10^{24} \text{ kg} \]
\[ R = r_m = 6345 \text{ km} \]
\[ g = \frac{G \text{M}_{\text{inside}}}{r_m^2} = \frac{(6.6 \times 10^{11})(5.94 \times 10^{24})}{(6345 \times 10^3)^2} \]
\[ g = 9.85 \text{ m/s}^2 \]

(c) \[ g = g_s \left(1 - \frac{D}{R_E}\right) \]
\[ = 9.83 \left(1 - \frac{25}{6370}\right) \]
\[ g = 9.79 \text{ m/s}^2 \]

2) \[ p_E = \frac{M_E}{V_E} = \frac{M_E}{\frac{4\pi R_E^3}{3}} \]

Mass inside D is:
\[ M_{\text{inside}} = \frac{4\pi}{3} p_E (R_E - D)^3 \]
\[ g = \frac{G M_{\text{inside}}}{r^2} \]
\[ = \frac{4\pi}{3} G p_E \frac{(R_E - D)^3}{(R_E - D)^2} \]
\[ = \frac{4\pi}{3} G \frac{M_E}{R_E^2} (R_E - D) = \frac{GM_E}{R_E^2} \left(1 - \frac{D}{R_E}\right) \]
5. a)

<table>
<thead>
<tr>
<th>Boundary</th>
<th>$\bar{r}$ (boundary, km)</th>
<th>$p &lt; (10^3 \text{ kg/m}^3)$</th>
<th>$p &gt; (10^3 \text{ kg/m}^3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Core / Mantle</td>
<td>3490</td>
<td>10.8</td>
<td>4.50</td>
</tr>
<tr>
<td>Mantle / Crust</td>
<td>6345</td>
<td>5.55</td>
<td>3.10</td>
</tr>
<tr>
<td>Crust / Air</td>
<td>6375</td>
<td>5.52</td>
<td>$\sim 0$</td>
</tr>
</tbody>
</table>

This is

$$\sum M_i \text{(inside)} \quad \frac{4\pi}{3} \bar{r}^3$$

This is

$$M_i \text{(just outside)} \quad \frac{4\pi}{3} \left( r_{\text{next boundary}}^3 - \bar{r}^3 \right)$$

b) The mass inside radius $r$ has two contributions, consisting of

i) mass inside $\bar{r} = \frac{4\pi}{3} p < r^3$

ii) mass between $\bar{r}$ and $r$:

$$= \frac{4\pi}{3} p_r (r^3 - \bar{r}^3)$$

$$g = \frac{G}{r^2} M_{\text{inside}} = \frac{G}{r^2} \left( \frac{4\pi}{3} p < r^3 + \frac{4\pi}{3} p_r (r^3 - \bar{r}^3) \right)$$
\[ g = G \left[ \frac{4\pi}{3} \left( P_1 - P_2 \right) \hat{r}^3 + \frac{4\pi}{3} P_2 \hat{r} \right] \]

so
\[ A = \frac{4\pi}{3} \left( P_1 - P_2 \right) \hat{r}^3 \quad B = \frac{4\pi}{3} P_2 \hat{r} \]

but we know from problem 4 that when \( P_1 = P_2 = P \)
\[ g = \frac{4\pi}{3} G P r \], so, when \( P \) uniform, \( A \) must be zero, and it is.

\[ g = \frac{G M_{\text{inside}}}{r^2} \]

\[ M_{\text{inside}} = 1.93 \times 10^{24} \text{ kg (core)} \]
\[ r = 3490 \text{ km (core)} \]

\[ g_c = \frac{(6.67 \times 10^{-11}) (1.93 \times 10^{24})}{(3490 \times 10^3)^2} \]
\[ g_c = 10.6 \text{ m/s}^2 \]

from \( r = 0 \) to \( r = r_c \) (core mantle boundary)
\[ M_{\text{inside}} = \frac{P_{\text{core}}}{3} \frac{4\pi}{3} r^3 = \frac{M_{\text{core}}}{3} \frac{4\pi}{3} r_c^3 = M_{\text{core}} \frac{r_c^3}{r^3} \]
\[ g(r) = \frac{G M_{\text{inside}}}{r^2} = \frac{G M_{\text{core}}}{r^2} \left( \frac{r_c^3}{r^3} \right) = \frac{G M_{\text{core}}}{r_c^2} \left( \frac{r}{r_c} \right) \]
\[ g(r) = g_c \left( \frac{r}{r_c} \right) = \left( 10.6 \text{ m/s}^2 \right) \left( \frac{r}{2490 \text{ km}} \right) \]
\[ g = G \left( \frac{A}{r^2} + Br \right) \]

\[ \frac{dg}{dF} = G \left( -\frac{2A}{r_{\text{min}}^3} + B \right) = 0 \]

\[ r_{\text{min}} = \left( \frac{2A}{B} \right)^{\frac{1}{3}} = \left( 2 \left[ \frac{p_\text{<}}{p_\text{>}} - 1 \right] \right)^{\frac{1}{3}} r_\text{H} \]

\[ g_{\text{min}} = G \left( \frac{A}{(2A)^{\frac{2}{3}}} + B \left( \frac{2A}{B} \right)^{\frac{1}{3}} \right) \]

\[ = G \left( \frac{1}{2^{\frac{2}{3}}} (B^2A)^{\frac{1}{3}} + 2^{\frac{2}{3}} (B^2A)^{\frac{1}{3}} \right) \]

\[ g_{\text{min}} = G \left( \frac{1}{2^{\frac{2}{3}}} + 2^{\frac{2}{3}} \right) (A^2B^2)^{\frac{1}{3}} \]

\[ = \frac{4\pi G}{3} \left( \frac{1}{2^{\frac{2}{3}}} + 2^{\frac{2}{3}} \right) \left( \frac{p_\text{<} - p_\text{>}}{\bar{p}_\text{>}} \right)^{\frac{2}{3}} r_\text{H}^{\frac{1}{3}} \]

\[ g_{\text{min}} = \frac{4\pi G}{3} \bar{p}_\text{>} \tilde{r} \left( \frac{1}{2^{\frac{2}{3}}} + 2^{\frac{2}{3}} \right) \left( \frac{p_\text{<}}{p_\text{>}} - 1 \right)^{\frac{1}{3}} \]

\[ \frac{d^2g}{dr^2} = G \left( \frac{6A}{r_{\text{min}}^4} \right) \quad \text{above is a minimum; need} \quad \frac{p_\text{<}}{p_\text{>}} > 1, \quad \text{higher density inside.} \]

\[ for \quad r_{\text{min}} > \tilde{r}, \quad \text{need} \]

\[ (2 \left[ \frac{p_\text{<}}{p_\text{>}} - 1 \right])^{\frac{1}{3}} > 1 \quad \Rightarrow \quad \frac{p_\text{<}}{p_\text{>}} > \frac{3}{2} \]
numerically
\[ r_{\text{min}} = \left(2 \cdot \left(\frac{10.8}{4.50} - 1\right)\right)^{\frac{1}{3}}, \quad 3490 \text{ km} \]
\[ r_{\text{min}} = 4932 \text{ km} \]
\[ g_{\text{min}} = \frac{4\pi}{3} \cdot (6.673 \times 10^{-11}) \cdot (4.50 \times 10^3) \cdot (3490 \times 10^3) \]
\[ \times \left(\frac{1}{2^{2/3}} + 2^{1/3}\right) \left(\frac{10.8}{4.50} - 1\right)^{1/3} \]
\[ g_{\text{min}} = 9.30 \text{ m/s}^2 \]

(e) core: \[ g(r) = \left(10.6 \frac{\text{m}}{\text{s}^2}\right) \left(\frac{r}{3490 \text{ km}}\right) \]

mantle: \[ g(r) = \frac{4\pi}{3} G \left(p - p_0\right) \cdot \left(\hat{r} \cdot \hat{r}\right) \frac{(r)}{r^2} + \frac{4\pi}{3} G p \cdot \hat{r} \left(\frac{r}{F}\right) \]
\[ = \frac{4\pi}{3} \cdot (6.673 \times 10^{-11}) \cdot (10.8 - 4.5) \cdot 10^3 \cdot (3490 \cdot 10^3) \]
\[ = 6.19 \text{ m/s}^2 \]
\[ = \frac{4\pi}{3} \cdot (6.673 \times 10^{-11}) \cdot (4.5 \cdot 10^3) \cdot (3490 \cdot 10^3) \]
\[ = 4.39 \text{ m/s}^2 \]
\[ g(r) = (6.19 \frac{\text{m}}{\text{s}^2}) \left(\frac{3490 \text{ km}}{r}\right) + (4.39 \frac{\text{m}}{\text{s}^2}) \left(\frac{r}{3490 \text{ km}}\right) \]

crust: \[ \frac{4\pi}{3} G \left(p - p_0\right) \cdot \hat{r} = \frac{4\pi}{3} \cdot (6.673 \times 10^{-11}) \cdot (5.55 - 3.10) \cdot 10^3 \cdot (6345 \cdot 10^3) \]
\[ = 4.24 \text{ m/s}^2 \]
Effect of core, mantle, crust density on acceleration of gravity in and above the Earth

\[ g \text{ (m/s}^2\text{)} \]

Radius (km)

\[ g, \text{ acceleration due to gravity} \]
\[ \frac{4\pi}{3} GR_7^2 = \frac{4\pi}{3} (6.673 \times 10^{-11}) (3.10 \times 10^3) (6375 \times 10^3) \]
\[ = 5.50 \text{ m/s}^2 \]
\[ g(r) = \left(4.34 \frac{\text{m}}{\text{s}^2}\right) \left(\frac{6345 \text{ km}}{r}\right) + \left(5.50 \frac{\text{m}}{\text{s}^2}\right) \left(\frac{r}{6345 \text{ km}}\right) \]

\text{air:} \quad g = \frac{GM_E}{r^2} = \frac{GM_E}{R_E^2} \cdot \left(\frac{R_E}{r}\right)^2

\[ g_\text{s} \]
\[ g(r) = 9.83 \frac{\text{m}}{\text{s}^2} \left(\frac{6370 \text{ km}}{r}\right) \]

\[ \text{Fig. 2.26} \]
\[ P_E = \frac{M_E}{\frac{4\pi}{3} R_E^3} \]
\[ M_{\text{inside}} = \frac{4\pi}{3} P_E R_E^3 \]
\[ F = G \frac{M_{\text{inside}} M}{R^2} \]
\[ = G \frac{4\pi}{3} P_E R_E^3 \cdot m \]
\[ = G \frac{M_E}{R_E^2} \cdot \left(\frac{R}{R_E}\right) \cdot m \]
\[ q \]
\[ F = \left( \frac{R}{R_E} \right) mg = \left( \frac{mg}{R_E} \right) R \]

\[ \text{direction is } \vec{F} = -kR \hat{r} \]

\[ w = \sqrt{\frac{k}{m}} = \sqrt{\frac{mg}{mR_E}} = \sqrt{\frac{g}{R_E}} \]

\[ T = \frac{2\pi}{w} = 2\pi \sqrt{\frac{R_E}{g}} \]

\[ = 2\pi \sqrt{\frac{6370 \times 10^3}{9.83}} \]

\[ T = 5060 \text{ s} = 84.3 \text{ min} = 1 \text{ hour 24 minutes} \]

**Low Earth Orbit**

\[ F_{\text{net}} = ma \]

\[ - \frac{GM_{\text{Earth}} m}{R_E^2} = m a = \frac{\Delta v^2}{R_E} \]

\[ \Delta v = \sqrt{g R_E} \]

\[ T = \frac{2\pi R_E}{\Delta v} = \frac{2\pi R_E}{\sqrt{g R_E}} = \frac{2\pi \sqrt{R_E}}{g} = \text{same!} \]
\[ -y = -y_1 - y_2 \]

Point \( a \) - no mass,

\[ k_1 y_1 - k_2 y_2 = 0 \]

\[ y_1 = \frac{k_2}{k_1} y_2 \]

So \( y = y_1 + y_2 = \left( \frac{k_2}{k_1} + 1 \right) y_2 \)

But on mass \( y_2 = \frac{y}{\left( \frac{k_2}{k_1} + 1 \right)} \)

\[ \uparrow k_2 y_2 \quad F_{\text{springs}} = k_2 y_2 = \frac{k_2}{\frac{k_2}{k_1} + 1} y \]

\[ \quad F_{\text{springs}} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}} y = k_a y \]

Or \( \frac{1}{k_a} = \frac{1}{k_1} + \frac{1}{k_2} \)

\[ k_a = \frac{k_1 k_2}{k_1 + k_2} \]
\[ \omega_a = \sqrt{\frac{k_a}{m}} = \sqrt{\frac{1}{m} \frac{k_1 k_2}{(k_1 + k_2)}} \]

\[ \omega_b = \sqrt{\frac{k_b}{m}} = \sqrt{\frac{k_1 + k_2}{m}} \]

(b) \[ F = (k_1 + k_2) y = k_b y \]

\[ k_b = k_1 + k_2 \]

**Hint:** \( k_1 = k_2 = k \)

\[ \omega_a = \sqrt{\frac{1}{m} \frac{k^2}{k+k}} = \sqrt{\frac{k}{2m}} \checkmark \]

\[ \omega_b = \sqrt{\frac{1}{m} \cdot 2k} = \sqrt{2 \frac{k}{m}} \checkmark \]