Discussion / Generalization

(some discussion in text, pp 106-7, 113-122)

$$\underset{\sim}{U}(\theta\hat{y}) = e^{\frac{-i\underset{\sim}{\sigma}_y\theta}{2}} = \cos\frac{\theta}{2} - i\underset{\sim}{\sigma}_y \sin\frac{\theta}{2} \doteq \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

rotates an existing spin state $|\psi\rangle$ by an angle θ about the y axis.

More generally, if you want to rotate about an arbitrary direction $\hat{n} = n_x\hat{x} + n_y\hat{y} + n_z\hat{z}$ with $1 = \sqrt{n_x^2+n_y^2+n_z^2}$, then define

$$\hat{n}\cdot\underset{\sim}{\vec{\sigma}} = n_x\underset{\sim}{\sigma}_x + n_y\underset{\sim}{\sigma}_y + n_z\underset{\sim}{\sigma}_z \doteq \begin{pmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{pmatrix}$$

vector of operators
$(\underset{\sim}{\sigma}_x, \underset{\sim}{\sigma}_y, \underset{\sim}{\sigma}_z)$

note:
$$(\hat{n}\cdot\underset{\sim}{\vec{\sigma}})^2 = (n_x\underset{\sim}{\sigma}_x + n_y\underset{\sim}{\sigma}_y + n_z\underset{\sim}{\sigma}_z)(n_x\underset{\sim}{\sigma}_x + n_y\underset{\sim}{\sigma}_y + n_z\underset{\sim}{\sigma}_z)$$
$$= n_x^2\cancel{\underset{\sim}{\sigma}_x^2}^{\,1} + n_y^2\cancel{\underset{\sim}{\sigma}_y^2}^{\,1} + n_z^2\cancel{\underset{\sim}{\sigma}_z^2}^{\,1}$$
$$+ n_xn_y\underbrace{\cancel{(\underset{\sim}{\sigma}_x\underset{\sim}{\sigma}_y + \underset{\sim}{\sigma}_y\underset{\sim}{\sigma}_x)}}_{\to 0} + n_xn_z\underbrace{\cancel{(\underset{\sim}{\sigma}_x\underset{\sim}{\sigma}_z + \underset{\sim}{\sigma}_z\underset{\sim}{\sigma}_x)}}_{\to 0}$$
$$+ n_yn_z\underbrace{\cancel{(\underset{\sim}{\sigma}_y\underset{\sim}{\sigma}_z + \underset{\sim}{\sigma}_z\underset{\sim}{\sigma}_y)}}_{\to 0}$$
$$= (n_x^2+n_y^2+n_z^2)\cdot\underset{\sim}{\mathbb{1}} = \underset{\sim}{\mathbb{1}}$$

so,
$$e^{\frac{-i(\vec{n}\cdot\underset{\sim}{\vec{\sigma}})\theta}{2}} = \cos\frac{\theta}{2} - i(\vec{n}\cdot\underset{\sim}{\vec{\sigma}})\sin\frac{\theta}{2} = \underset{\sim}{U}(\theta\hat{n})$$

describe rotations about arbitrary directions by amount θ

- they are unitary: $\underset{\sim}{U}^{-1}(\theta\hat{n}) = \underset{\sim}{U}^{\dagger}(\theta\hat{n})$
- they have determinant 1

$$\begin{vmatrix} \cos\frac{\theta}{2} - in_z\sin\frac{\theta}{2} & (-in_x - n_y)\sin\frac{\theta}{2} \\ (-in_x+n_y)\sin\frac{\theta}{2} & \cos\frac{\theta}{2} + in_z\sin\frac{\theta}{2} \end{vmatrix} = \cos^2\frac{\theta}{2} + n_z^2\sin^2\frac{\theta}{2} + n_x^2\sin^2\frac{\theta}{2} + n_y^2\sin^2\frac{\theta}{2} = 1$$

- form a group known as SU(2)
  - S → special means determinant 1
  - U → U means unitary
  - 2 → 2 means 2x2 matrices.

Precession of a Moment, Quantum Mechanically

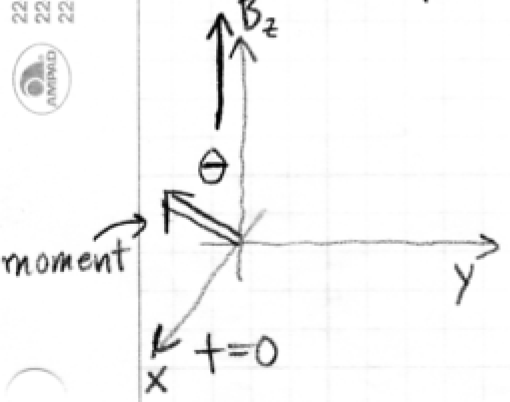
classically, $U = -\vec{m} \cdot \vec{B}$

make these operators... $\underset{\sim}{\vec{m}} = \frac{gq}{2m} \underset{\sim}{\vec{S}}$

$$\underset{\sim}{H} = \frac{-gq}{2m} \vec{B} \cdot \underset{\sim}{\vec{S}} = \frac{-gq\hbar}{4m}\left(B_x \underset{\sim}{\sigma}_x + B_y \underset{\sim}{\sigma}_y + B_z \underset{\sim}{\sigma}_z\right)$$

$$\doteq -\frac{gq\hbar}{4m}\begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}$$

Set up problem like classical case



initial state $\doteq \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix} = \psi(0)$

Schrödinger equation:

$$i\hbar \frac{d}{dt}\psi = \underset{\sim}{H}\psi = \frac{-gq\hbar}{4m} B_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\psi$$

eigenstates: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ e.v. $-\frac{gq\hbar}{4m}B_z = -\frac{\hbar}{2}\omega_p$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ev $+$ " $= +\frac{\hbar}{2}\omega_p$

so the state $\begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix}$ evolves to $\begin{pmatrix} e^{+i\omega_p t/2} \cos\theta/2 \\ e^{-i\omega_p t/2} \sin\theta/2 \end{pmatrix}$

$$\langle S_x(t)\rangle = \left(e^{-i\omega_p t/2}\cos\frac{\theta}{2} \quad e^{+i\omega_p t/2}\sin\frac{\theta}{2}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{i\omega_p t/2}\cos\theta/2 \\ e^{-i\omega_p t/2}\sin\theta/2 \end{pmatrix} \times \frac{\hbar}{2}$$

$$= \left(e^{-i\omega_p t} + e^{+i\omega_p t}\right)\sin\frac{\theta}{2}\cos\frac{\theta}{2} \times \frac{\hbar}{2} = \underline{\frac{\hbar}{2}\sin\theta\cos\omega_p t}$$

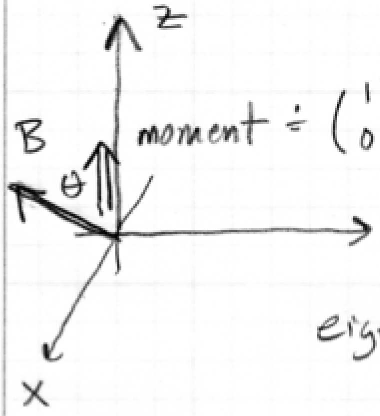
$$\langle S_y(t)\rangle = \qquad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= i\left(-e^{-i\omega_p t} + e^{+i\omega_p t}\right)\sin\frac{\theta}{2}\cos\frac{\theta}{2} \times \frac{\hbar}{2} = \underline{\frac{\hbar}{2}\sin\theta\sin\omega_p t}$$

$$\langle S_z(t)\rangle = \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right) \times \frac{\hbar}{2} = \underline{\frac{\hbar}{2}\cos\theta}$$

Related problem



moment $\doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ initially

$$\underset{\sim}{B} = B\cos\theta\,\underset{\sim}{\sigma}_z + B\sin\theta\,\underset{\sim}{\sigma}_x$$

$$= B\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

$$\underset{\sim}{H} = -\frac{gq\hbar}{4m} B \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\sin\theta \end{pmatrix}$$

eigenstates are: $\begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix}$ e.v. $-\frac{gq\hbar}{4m} B = -\frac{\hbar}{2}\omega_p$

$\begin{pmatrix} -\sin\theta/2 \\ \cos\theta/2 \end{pmatrix}$ e.v. $+\frac{gq\hbar}{4m} B = +\frac{\hbar}{2}\omega_p$

at $t=0$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \cos\frac{\theta}{2}\begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix} - \sin\frac{\theta}{2}\begin{pmatrix} -\sin\theta/2 \\ \cos\theta/2 \end{pmatrix}$$

(project initial state on to eigenstates)

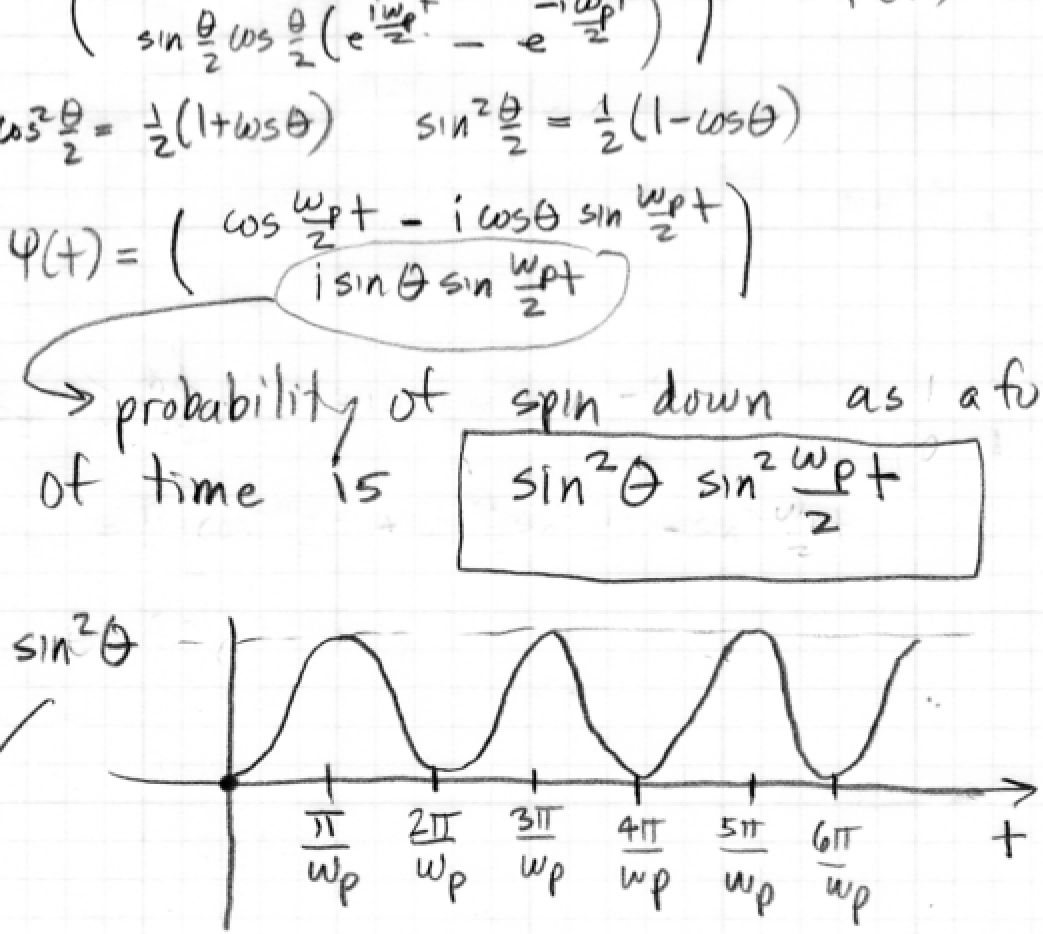
time development:

$$\begin{pmatrix} \cos^2\frac{\theta}{2} e^{\frac{i\omega_p t}{2}} + \sin^2\frac{\theta}{2} e^{-\frac{i\omega_p t}{2}} \\ \sin\frac{\theta}{2}\cos\frac{\theta}{2}\left(e^{\frac{i\omega_p t}{2}} - e^{-\frac{i\omega_p t}{2}}\right) \end{pmatrix} = \psi(t)$$

$$\cos^2\frac{\theta}{2} = \frac{1}{2}(1+\cos\theta) \qquad \sin^2\frac{\theta}{2} = \frac{1}{2}(1-\cos\theta)$$

$$\psi(t) = \begin{pmatrix} \cos\frac{\omega_p t}{2} - i\cos\theta\sin\frac{\omega_p t}{2} \\ i\sin\theta\sin\frac{\omega_p t}{2} \end{pmatrix}$$

probability of spin down as a function of time is $\boxed{\sin^2\theta\,\sin^2\frac{\omega_p t}{2}}$



when $\sin\theta = 1$, $\theta = \pi/2$, gets to 100%!