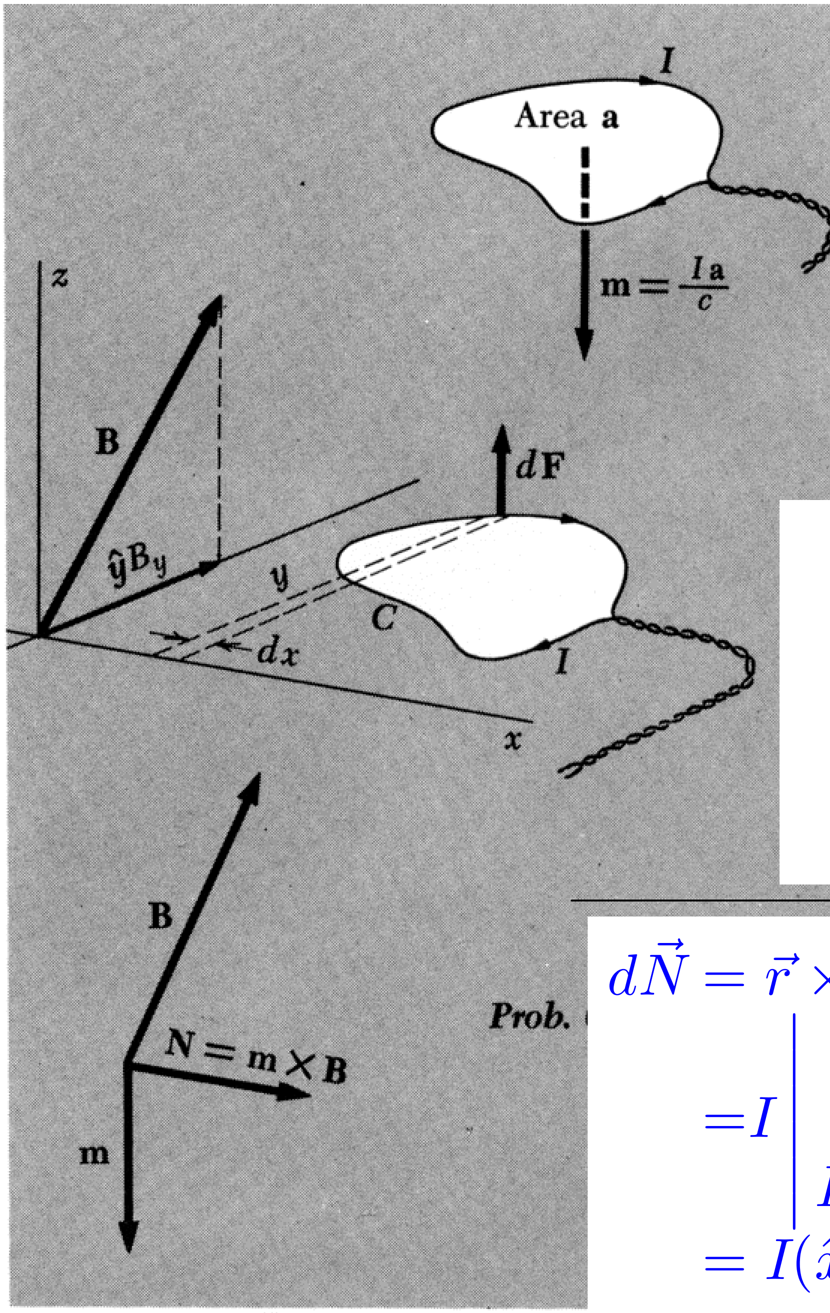
Force, Torque on a Current Loop in a Uniform Magnetic Field



$\vec{B}$ uniform everywhere

Choose $\vec{B}$ to be in the y-z plane

$$\vec{B} = B_y\hat{y} + B_z\hat{z}$$

line element of of path C: $d\vec{\ell}$

$$d\vec{F} = I d\vec{\ell} \times \vec{B}$$
$$= I[dx\hat{x} + dy\hat{y}] \times [B_y\hat{y} + B_z\hat{z}]$$
$$= I[dx(B_y\hat{z} - B_z\hat{y}) + dyB_z\hat{x}]$$

but $0 = \int_C dx = \int_C dy$

$0 = \int_C d\vec{F}$ .... no net force

$$d\vec{N} = \vec{r} \times d\vec{F}$$
$$= I \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ B_z dy & -B_z dx & B_y dx \end{vmatrix}$$
$$= I(\hat{x}B_y y dx - \hat{y}B_y x dx - \hat{z}B_z[xdx + ydy])$$

$\int_C x dx = \int_C y dy = 0$; but $\int_C y dx = a(\text{area})$

$\vec{N} = \int_C d\vec{N} = IaB_y\hat{x} = \vec{m} \times \vec{B}; N = mB\sin\theta$

Orientation: figure shows $\int_C y dx > 0$
and $\vec{m}$ must point downward
for cross product to yield $\int_C d\vec{N}$

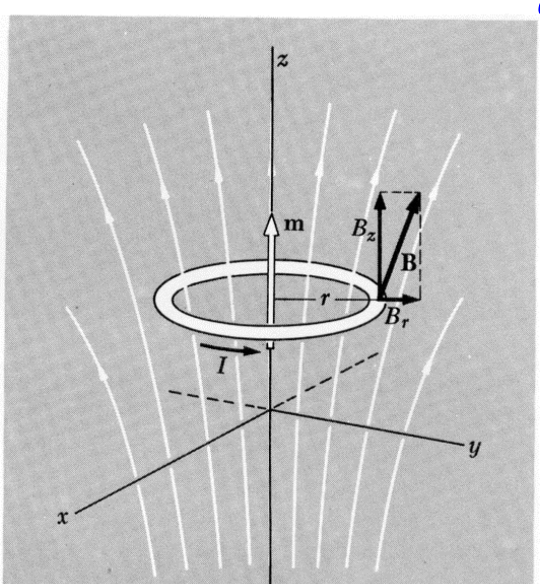
Energy of a Dipole in a Magnetic Field

$$\Delta U = \int_{\theta_1}^{\theta_2} d\theta N(\theta) = mB \int_{\theta_1}^{\theta_2} d\theta \sin\theta$$
$$= -mB[\cos\theta_2 - \cos\theta_1], \text{ so}$$
$$U = -mB\cos\theta = -\vec{m}\cdot\vec{B}$$

Dipole tends to **align with** magnetic field

Force on a Dipole in a Non-Uniform Magnetic Field

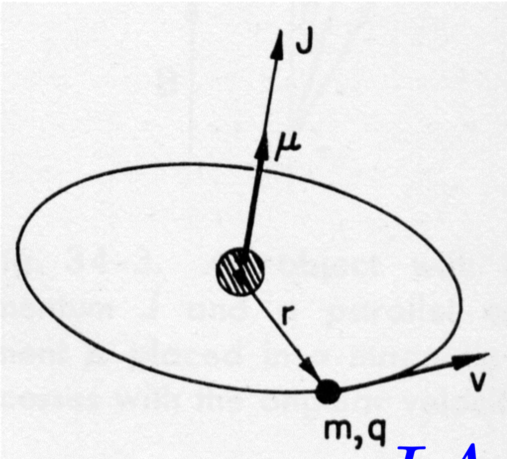
$$\vec{F} = -\vec{\nabla}(-\vec{m}\cdot\vec{B})$$
$$\text{when } \vec{m} \text{ is along } \hat{z},$$
$$0 = \partial B_z/\partial x = \partial B_z/\partial y$$
$$\text{(center of a loop)}$$
$$F_z = m\frac{\partial B_z}{\partial z}$$



Force develops from the component of the field **perpendicular** to z !

$$\vec{\nabla}\cdot\vec{B} = 0$$

Magnetic Moment Related to Angular Momentum



$$\vec{J} \rightarrow \vec{L}, \vec{\mu} \rightarrow \vec{m}$$

$$m = IA = I \times \pi r^2$$
$$= \left[\frac{qvT}{2\pi r}\frac{1}{T}\right] \times \pi r^2 = \frac{1}{2} qvr$$
$$= \frac{q}{2m} mvr = \frac{q}{2m} L$$

$$\text{charge } q = \int_V d^3x \rho_q(\vec{x}) \quad \text{Mass } M = \int_V d^3x \rho_M(\vec{x})$$

$$\text{key assumption:} \quad \rho_q(\vec{x}) = \frac{q}{M} \rho_M(\vec{x}) \quad \textit{not necessarily so!}$$

$$\text{oriented area:} \quad d\vec{a} = \frac{1}{2} \vec{r} \times d\vec{\ell}$$

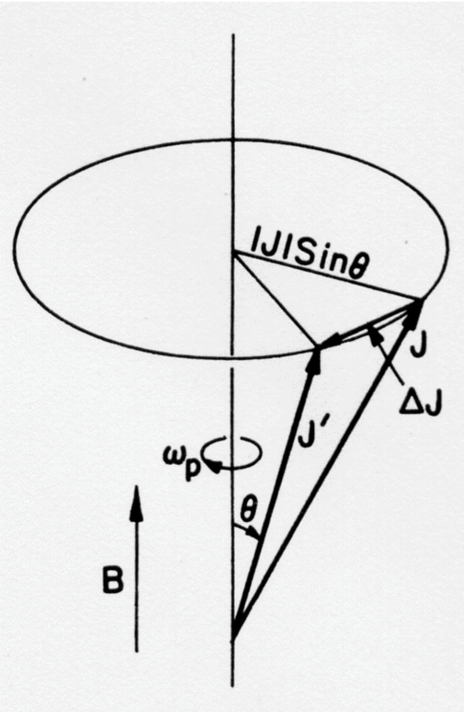
$$\text{current:} \quad I d\vec{\ell} = \rho_q(\vec{x}) \vec{v} A d\ell$$

$$\text{moment:} \quad d\vec{m} = I d\vec{a} = \frac{1}{2} \vec{r} \times \left([\rho_q(\vec{x}) A d\ell] \vec{v} \right)$$
$$= \frac{q}{2M} \vec{r} \times \left([\rho_M(\vec{x}) A d\ell] \vec{v} \right)$$
$$= \frac{q}{2M} \vec{r} \times d\vec{p} = \frac{q}{2M} d\vec{L}$$

g is a fudge factor

$$\text{generally,} \quad \vec{m} = g \frac{q}{2M} \vec{s} \quad s \text{ is spin}$$

Spin Precession in Magnetic Field



$$\vec{J} \to \vec{s}$$

$$\frac{d\vec{s}}{dt} = \vec{m} \times \vec{B}$$

$$= g\frac{q}{2M}\vec{s} \times \vec{B}$$

$$\Delta\phi = \omega_p \Delta t = \frac{|\Delta\vec{s}|}{|\vec{s}|\sin\theta} = g\frac{q}{2M}B\,\Delta t$$

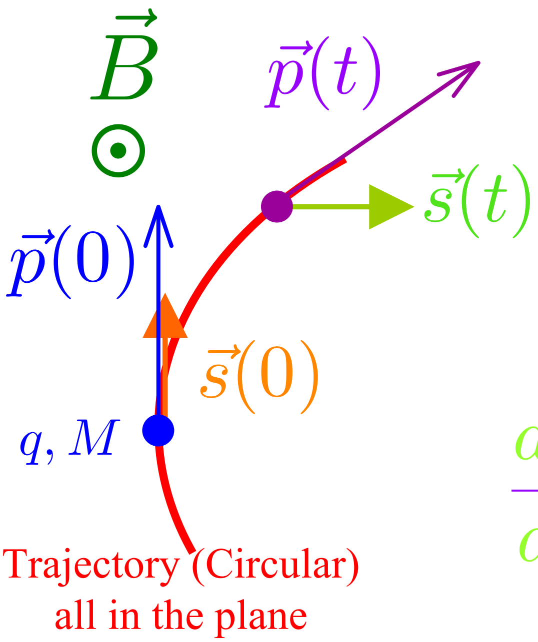
$$\omega_p = g\frac{q}{2M}B$$

$$s_x(t) = s\sin\theta\cos\omega_p t$$

$$s_y(t) = s\sin\theta\sin\omega_p t$$

$$s_z(t) = s\cos\theta \quad \text{...stable in time}$$

Physical Picture of g=2



$$\frac{d\vec{p}}{dt} = q\vec{v} \times \vec{B} = \frac{q}{M}\vec{p} \times \vec{B}$$

$$\omega_1 = \frac{|\Delta\vec{p}|}{|\vec{p}|\Delta t} = \frac{q}{M}|\vec{B}|$$

$$\frac{d\vec{s}}{dt} = \vec{m} \times \vec{B} = g\frac{q}{2M}\vec{s} \times \vec{B}$$

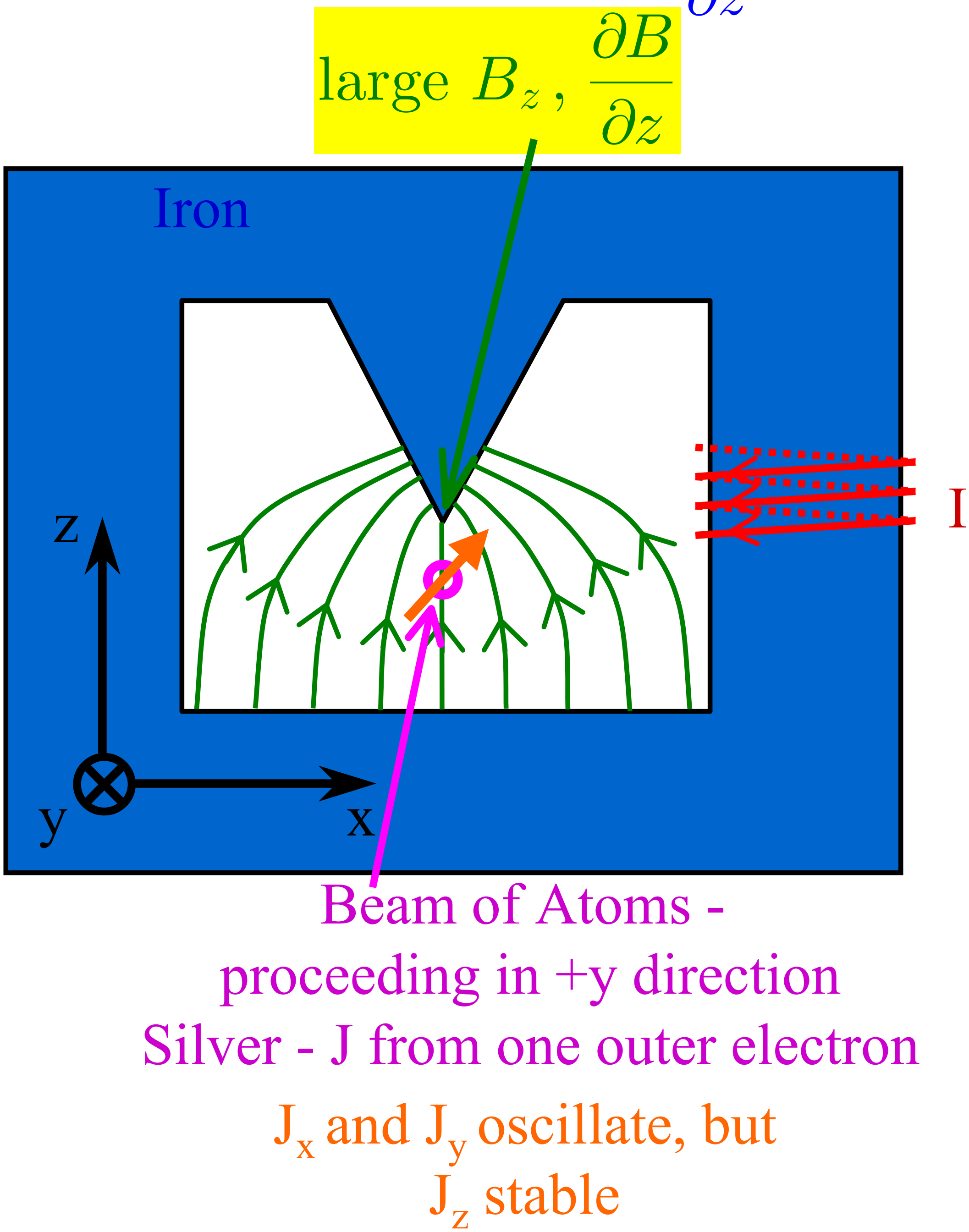
$$\omega_2 = \frac{|\Delta\vec{s}|}{|\vec{s}|\Delta t} = g\frac{q}{2M}|\vec{B}|$$

$$\omega_1 = \omega_2 \quad \rightarrow \quad g = 2$$

- Empirically very nearly true for charged leptons: e, μ, τ
- Deviations from g=2 due to higher order quantum corrections… one of the most interesting areas of experimental work… `g-2’ experiments

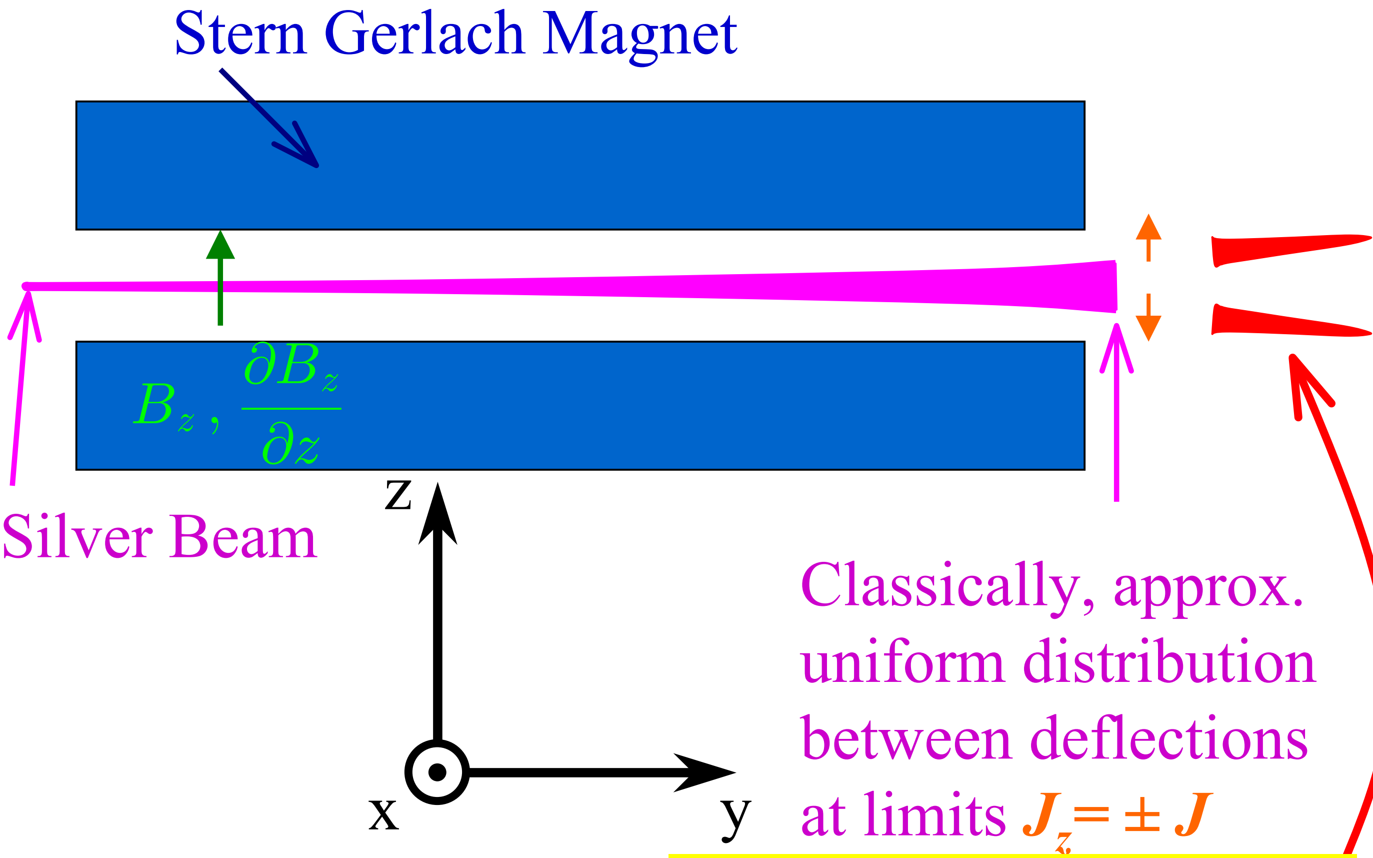
Stern-Gerlach Experiment (1921) : Quantum Measurement of spin-½

Experiment uses a magnet that has both a large B_z and a large $\frac{\partial B_z}{\partial z}$.



$$\langle F_z \rangle = \left[g \frac{q}{2M} \frac{\partial B_z}{\partial z} \right] \times J_z$$

Surprise: Classical Expectation Not Right at All



Quantum Mechanically, beam splits into *two*, with: $J_z = \pm \frac{\hbar}{2}$

Electron *spin* causes J_z, and s_z has eigenvalues of $\pm\hbar/2$:

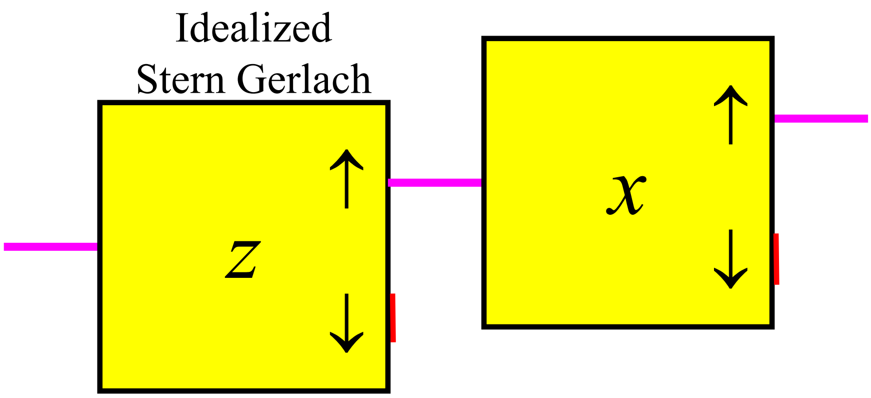
$$s_z \doteq \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \equiv \frac{\hbar}{2}\sigma_z; \quad \text{note} \quad \sigma_z^2 = 1$$

$$\sigma_z = \sigma_3; \quad \text{"Pauli Matrix"}$$

More Quantum Mechanics, and s_x

$$\uparrow_z \quad \Longrightarrow \quad |\uparrow_z\rangle \doteq \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\downarrow_z \quad \Longrightarrow \quad |\downarrow_z\rangle \doteq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



Simultaneous eigenstate of s_x and s_z?

s_x must also be represented by a 2 by 2 hermitian matrix:

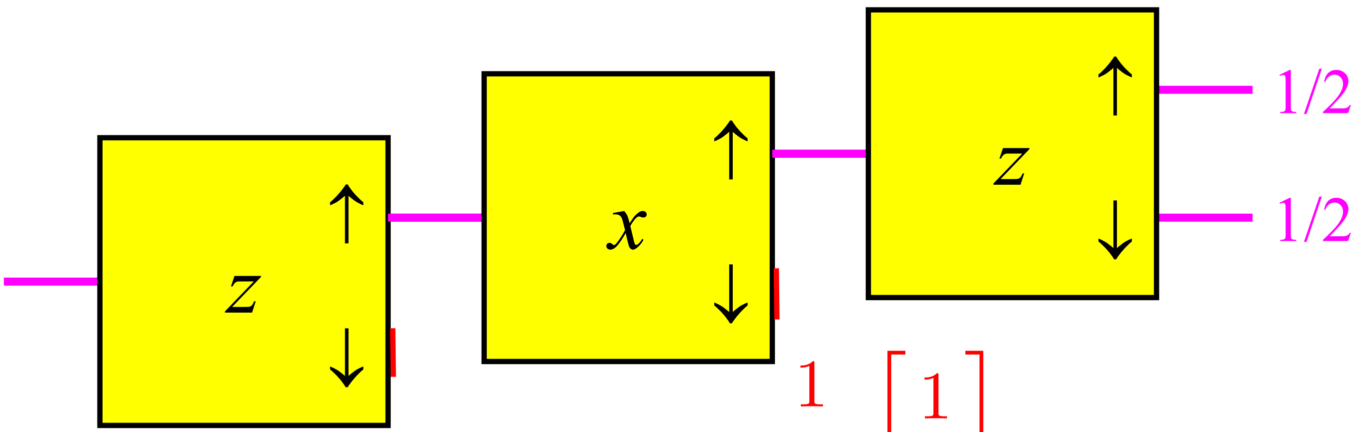
$$s_x = \frac{\hbar}{2}\sigma_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; s_y = \frac{\hbar}{2}\sigma_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

The x Stern Gerlach measures eigenvalue of s_x, and after measurement, the state is an eigenstate.

$$+\frac{\hbar}{2}: \quad \uparrow_x \quad \Longrightarrow \quad |\uparrow_x\rangle \doteq \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$-\frac{\hbar}{2}: \quad \downarrow_x \quad \Longrightarrow \quad |\downarrow_x\rangle \doteq \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Three Measurements, Commutation



$$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

Measurement of s_x disturbs the eigenstate of s_z.

$$\left[s_z, s_x\right] = \frac{\hbar^2}{4}\left[\sigma_z, \sigma_x\right]$$

$$\left[\sigma_z, \sigma_x\right] = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} - \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} = 2i\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$$

$$\left[\sigma_z, \sigma_x\right] = \left[\sigma_3, \sigma_1\right] = 2i\sigma_y = 2i\sigma_2$$

$$\frac{1}{2i}\left[\sigma_j, \sigma_k\right] = \epsilon_{jkl}\sigma_l$$

$$\epsilon_{123} = \epsilon_{312} = \epsilon_{231} = 1$$

$$\epsilon_{321} = \epsilon_{132} = \epsilon_{213} = -1$$

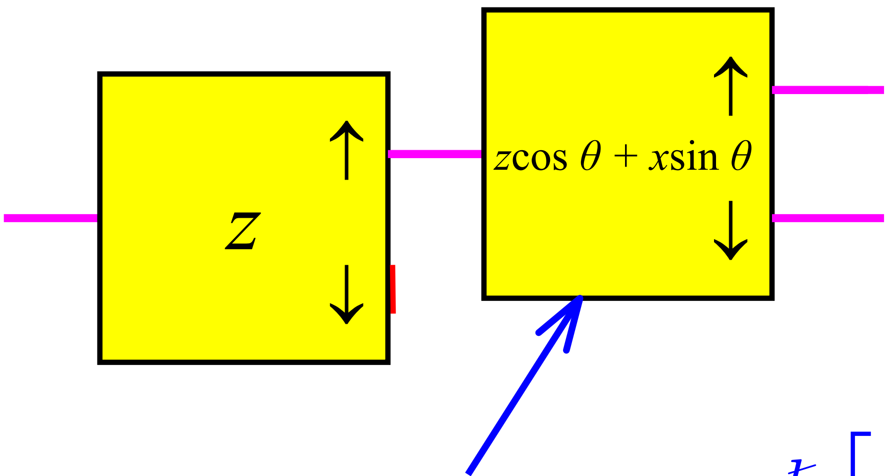
all other $\epsilon_{jkl} = 0$

$$\frac{1}{2}\left\{\sigma_j, \sigma_k\right\} = \delta_{jk}$$

$\{\,\}$ denotes anticommutator

Commutation relations are mathematically fundamental, valid for all angular momentum operators in quantum mechanics.
Anticommutation valid for spin-1/2 only.

Arbitrary Angle Stern-Gerlach



$$\begin{bmatrix} \cos\theta/2 \\ \sin\theta/2 \end{bmatrix}$$

$$\begin{bmatrix} -\sin\theta/2 \\ \cos\theta/2 \end{bmatrix}$$

$$s_z \cos\theta + s_x \sin\theta = \frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

eigenvalues:

$$\begin{vmatrix} \cos\theta - \lambda & \sin\theta \\ \sin\theta & -\cos\theta - \lambda \end{vmatrix} = \lambda^2 - \cos^2\theta - \sin^2\theta = \lambda^2 - 1 = 0$$

$$\lambda = \pm 1 \Longrightarrow \text{spin e.v.} = \pm\hbar/2$$

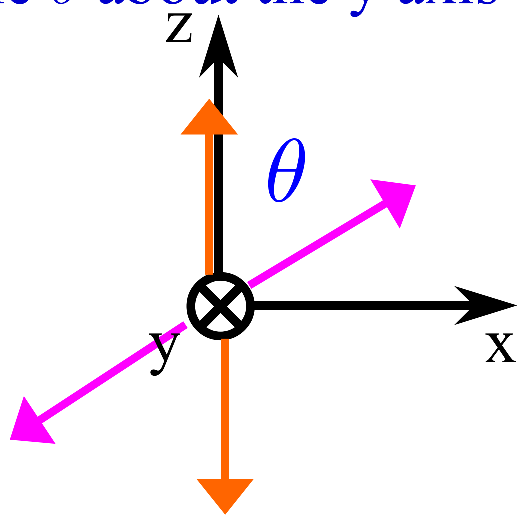
$+\hbar/2:$

$$\left\{\begin{bmatrix} \cos\theta - 1 & \sin\theta \\ \sin\theta & -\cos\theta - 1 \end{bmatrix} = 2\begin{bmatrix} -\sin^2\theta/2 & \sin\theta/2\cos\theta/2 \\ \sin\theta/2\cos\theta/2 & -\cos^2\theta/2 \end{bmatrix}\right\}\begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$\frac{a}{b} = \frac{\cos\theta/2}{\sin\theta/2}(+\hbar/2); = -\frac{\sin\theta/2}{\cos\theta/2}(-\hbar/2)$$

Transformations that Describe Rotations

$U(\theta\hat{y})$: the operator which acts on a state in the 2 dimensional spin space and pushes it actively by an angle θ about the y axis



$$U(\theta\hat{y}) \doteq \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

$$U^{-1}(\theta\hat{y}) = U(-\theta\hat{y}) \doteq \begin{bmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \doteq U^{\dagger}(\theta\hat{y})$$

Unitary

$$U(\theta\hat{y}) = \cos\frac{\theta}{2} - i\sigma_y \sin\frac{\theta}{2}$$

$$= e^{\frac{-i\sigma_y\theta}{2}}$$

$$e^{\frac{-i\sigma_y\theta}{2}} = 1 - \frac{i\sigma_y\theta}{2} - \frac{1}{2!}\left[\frac{\sigma_y\theta}{2}\right]^2 + \frac{i}{3!}\left[\frac{\sigma_y\theta}{2}\right]^3 + \ldots$$

$$= 1 - \frac{1}{2!}\left[\frac{\theta}{2}\right]^2 + \frac{1}{4!}\left[\frac{\theta}{2}\right]^4 + \ldots - i\sigma_y\left(\frac{\theta}{2} - \frac{1}{3!}\left[\frac{\theta}{2}\right]^3 + \ldots\right)$$

$$= \cos\frac{\theta}{2} - i\sigma_y \sin\frac{\theta}{2}$$