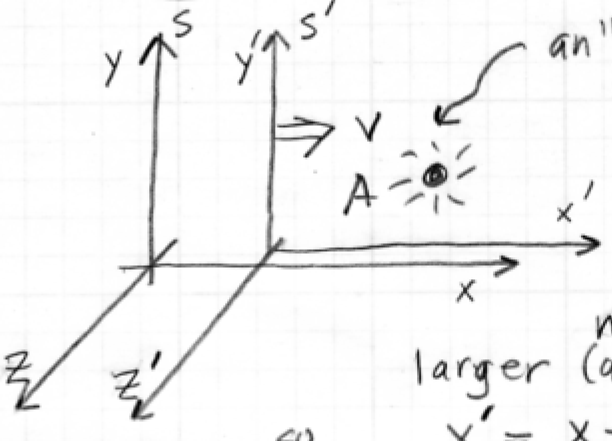
Relativistic Kinematics



an "event" in $S: (x,y,z,t)$ / $S': (x',y',z',t')$ } will use x_A, etc when needed

A

say, at $t=t'=0$, $x=x'$, $y=y'$, $z=z'$.

non-relativistically, x is larger (as drawn) than x' for $v>0$, so, $x' = x - vt$, while $y'=y$, $z'=z$, $t'=t$

RELATIVISTICALLY, in order that no observer sees a speed that exceeds that of light, something "gives"... $t \neq t'$, although $y'=y$ and $z'=z$ (those are the components $\perp$ to the direction of motion)

$$\gamma \equiv \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \geq 1 = \frac{1}{\sqrt{1-\beta^2}} \quad \beta = \frac{v}{c}$$

$$x' = \gamma(x - vt)$$
$$t' = \gamma\left(t - \frac{v x}{c^2}\right)$$

mnemonic: $\propto v$ like above, $1/c^2$ to get dimensions

① Simultaneity can depend on frame

Consider second event, B, where $\underbrace{t_A = t_B}_{\text{defines simultaneity}}$ but $x_A \neq x_B$

$$t_A' = \gamma\left(t_A - \frac{v}{c^2}x_A\right)$$
$$t_B' = \gamma\left(t_B - \frac{v}{c^2}x_B\right)$$

$$\Delta t' = t_A' - t_B' = \frac{-v\gamma}{c^2}(x_A - x_B) \quad \text{now, } \neq 0$$

② Length Contraction

In ①, considered $\Delta t' = t_A' - t_B'$ for events that were simultaneous in S, now consider $\Delta x' = x_A' - x_B'$ (now in S):

$$x_A' = \gamma(x_A - vt_A)$$
$$x_B' = \gamma(x_B - vt_B)$$

$$\Delta x' = x_A' - x_B' = \gamma(x_A - x_B) = \gamma\Delta x \quad \text{or} \quad \Delta x = \frac{1}{\gamma}\Delta x' \quad \text{so } \Delta x \leq \Delta x'$$

This is "length contraction." it is important to remember that measurement of the length of a moving object is not easy, and the meaning is that the length of a moving object is the distance between simultaneous events that occur at the ends of the object (which is moving!). Those events will not be simultaneous in any other frame, particularly in the objects rest frame

(3) Time dilitation

Now, consider, for example, an unstable particle with mean life τ that is at $x'=y'=z'=0$ in S'. The first event is, say, the birth of the particle at $x'_A=y'_A=z'_A=t'_A=0$, and the second event is the decay of the particle at $x'_B=y'_B=z'_B=0$, $t'_B \simeq \tau$ (on average!). How long does the particle live in the frame S?

$$t_A = \gamma\left(t'_A - \frac{v}{c^2}x'_A\right) = 0$$

$$t_B = \gamma\left(t'_B - \frac{v}{c^2}\cancel{x'_B}\right) = \gamma t'_B = \gamma\tau$$

so $t_B - t_A = \gamma\tau$ and the particle appears to live longer in the frame S'. This effect is very important. The classic example is that of the muon, where

$$\tau_\mu \simeq 2\cdot 10^{-6}\ \mathrm{s} \quad \text{(a good one to commit to memory)}$$

In typical experiments, μ's are made that have $E_\mu \sim 10$ GeV, so, $\gamma \simeq \frac{E_\mu}{mc^2} \cong 100$, and in the lab frame,

$$t_B - t_A \simeq \gamma\tau_\mu \simeq 2\cdot 10^{-4}\ \mathrm{s}.$$

Such a μ appears to go essentially at the speed of light $v \simeq c$. The typical distance a μ like this goes is:

$$d = v(t_B - t_A) \simeq c\cdot\gamma\cdot\tau_\mu$$

$$\cong \left(1\,\frac{\text{foot}}{\text{nanosecond}}\right)\cdot 100\cdot(2\cdot 10^3\ \text{ns})$$

$$\simeq 2\cdot 10^5\ \text{ft} = 200\cdot 10^3\ \text{ft} \simeq 40\ \text{miles}$$

(4) velocity addition

$S' \rightarrow$ moves with x velocity component $u' = \frac{\Delta x'}{\Delta t'}$

$S \rightarrow$ 〃 $u = \frac{\Delta x}{\Delta t}$

but $\Delta x = \gamma(\Delta x' + v\Delta t')$ note + sign
$\Delta t = \gamma(\Delta t' + \frac{v}{c^2}\Delta x')$

velocity observed in S $\rightarrow$ $$u = \frac{\Delta x}{\Delta t} = \frac{\Delta x' + v\Delta t'}{\Delta t' + \frac{v}{c^2}\Delta x'} = \frac{\frac{\Delta x'}{\Delta t'} + v}{1 + \frac{v\Delta x'}{c^2 \Delta t'}} = \boxed{\frac{u' + v}{1 + \frac{u'v}{c^2}}}$$ velocity in u'

<u>note</u>: when $u' = c$, $u = \frac{c+v}{1+\frac{v}{c}} = c$ ← limiting velocity

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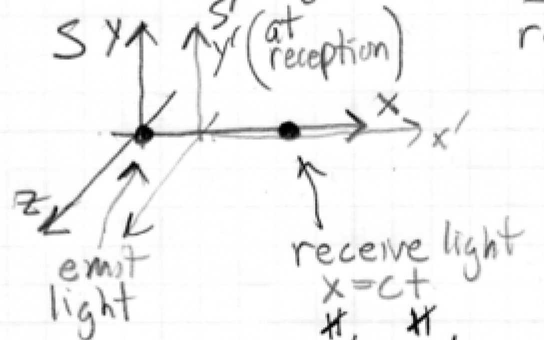
The invariant and 4-vectors

3-space: $r^2 = x^2 + y^2 + z^2$ ← invariant under Rotation

In 4-dimensions, what is like this?

Think of light beam <u>emitted</u> $x = y = z = t = 0 = x' = y' = z' = t'$

received $x = ct$ $x' = ct'$
$y = y' = z = z' = 0$



S+S' were coincident at t=t'=0

$$c^2t^2 - x^2 = 0 = c^2t'^2 - x'^2$$

- not all pairs of space + time points are connectable by a light beam.
- maybe $c^2t^2 - x^2$ still same in all frames...

$$c^2t'^2 - x'^2 = c^2\left(\gamma\left(t - \frac{v}{c^2}x\right)\right)^2 - \gamma^2(x - vt)^2$$
$$= c^2\gamma^2\left[t^2 - 2\frac{v}{c^2}xt + \frac{v^2}{c^4}x^2 - \frac{x^2}{c^2} + \cancel{2\frac{vtx}{c^2}} - \frac{v^2}{c^2}t^2\right]$$
$$= c^2\underbrace{\gamma^2\left(1 - \frac{v^2}{c^2}\right)}_{=1}\left(t^2 - \frac{x^2}{c^2}\right) = c^2t^2 - x^2 \quad \text{Yes.}$$

Homework: even $c^2t_1't_2' - x_1'x_2' = c^2t_1t_2 - x_1x_2$ (the invariant)

4-vector: $x^0 \equiv ct$ $x^1 \equiv x$ $x^2 \equiv y$ $x^3 \equiv z$

superscripts, <u>not</u> powers... "contravariant" indices.
ct is the "0th" component

4vector $x \equiv (x^0, x^1, x^2, x^3) = (x^0, \vec{x}) = (ct, \vec{x})$

x with nothing on top often a 4-vector, not a component

Generic 4-vector a (not necessarily coordinates of space + time)

$a = (a^0, \vec{a})$

Generally, GREEK indices are used for the components of a 4-vector, while LATIN indices are used for the 3-components of the 4-vector. Thus:

$$a^\mu \in (a^0, a^1, a^2, a^3)$$
$$a^i \in (a^1, a^2, a^3)$$

What is the difference between 4 random #'s and a 4-vector? It's the physics that leads to the correspondence between the 4-vector as seen in one frame (like S) and in another (S').

$$\left.\begin{aligned} a^{0\prime} &= \gamma(a^0 - \beta a^1) \\ a^{1\prime} &= \gamma(a^1 - \beta a^0) \\ a^{2\prime} &= a^2 \\ a^{3\prime} &= a^3 \end{aligned}\right\}$$

4 random #'s need not satisfy this relationship.

True also for a, if it is a 4-vector. Another notation is:

$$\Lambda^\mu_\nu = \begin{array}{c|cccc} & \nu=0 & 1 & 2 & 3 \\ \hline \mu=0 & & & & \\ 1 & & & & \\ 2 & & & & \\ 3 & & & & \end{array} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Leftarrow \underset{\sim}{\Lambda}$$

or $$a^{\mu\prime} = \sum_{\nu=0}^{3} \underbrace{\Lambda^\mu_\nu a^\nu}$$

Einstein's summation convention: any greek index that is repeated is to be summed over. Occasionally generalized to Latin indicies.

$$a^{\mu\prime} = \Lambda^\mu_\nu a^\nu$$

$\underset{\sim}{\Lambda}$ can describe boosts in arbitrary directions, and, rotations as well, or even the most general combination of the two.

The expression: $(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 =$

$$(x^{0\prime})^2 - (x^{1\prime})^2 - (x^{2\prime})^2 - (x^{3\prime})^2 \quad (*)$$

is described by introducing another matrix:

$$g_{\mu\nu} \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \text{"the metric"}$$

then: $(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = g_{\mu\nu} x^\mu x^\nu$

The "covariant" 4-vector is $x_\mu \equiv g_{\mu\nu} x^\nu$

$$= (x^0, -\vec{x})$$

& $x_\mu x^\mu = x'_\mu x^{\mu\prime}$ is the same as $(*)$

For a generic 4-vector a,

$a \cdot a \equiv a_\mu a^\mu \equiv a^2$ can be: > 0 (timelike 0th component "won")

$= a^{0\,2} - |\vec{a}|^2$

< 0 (spacelike 1-3 components "won")

same in all frames of reference

$= 0$ (lightlike; $\vec{x}$ & t "at the speed of light")

Better yet, for 2 4-vectors a & b,

$a \cdot b \equiv a_\mu b^\mu = a^0 b^0 - \vec{a} \cdot \vec{b}$ is also the same in all frames.

Energy & Momentum

$$4\text{-velocity} = \gamma \vec{v} \equiv \vec{\eta} \equiv \frac{\Delta \vec{x} \text{ in lab frame}}{\Delta \tau \text{ in rest frame}}$$

$$= \frac{\Delta \vec{x}}{\left(\frac{1}{\gamma}\Delta t\right)} = \gamma \vec{v} \rightarrow \gamma \frac{d\vec{x}}{dt}$$

0^{th} component: $\eta^0 = \dfrac{dx^0}{d\tau} = \gamma \dfrac{d(ct)}{dt} = \gamma c$

$\eta^\nu \eta_\nu \Rightarrow ?$ since $\eta^\nu = \frac{d}{d\tau} x^\nu$ 4-vector

expect it to be invariant ← frame independent

Compute: $\eta^\nu \eta_\nu = (\eta^0)^2 - |\vec{\eta}|^2 = \gamma^2 c^2 - \gamma^2 |\vec{v}|^2 = c^2\gamma^2(1-\frac{v^2}{c^2})$
$= c^2$

4-momentum

$p^\nu \equiv m\eta^\nu$ $\quad p^\nu p_\nu = m^2c^2$

the invariant for the 4-momentum is m^2c^2!

$p^0 = \gamma mc = \frac{E}{c} \rightarrow \frac{mc^2}{c}$ as $v \to 0$

$\vec{p} = \gamma m\vec{v} \longrightarrow m\vec{v}$ as $v \to 0$

$\gamma mc = \frac{1}{c} \cdot mc^2 \cdot \left(\frac{1}{\sqrt{1-\beta^2}}\right) \simeq \frac{1}{c} mc^2 \cdot \left(1 + \frac{1}{2}\beta^2 + \frac{3}{8}\beta^4 + \ldots\right.$

$= \frac{1}{c}\left[mc^2 + \frac{1}{2}mv^2 + \frac{3}{8}mv^2 \cdot \frac{v^2}{c^2} + \ldots\right]$

rest energy ↑ $\quad$ kinetic energy ↑ $\quad$ relativistic correction ↑

m=0 particles

Are never at rest (like photon)

→ no mc^2

$p^\nu p_\nu = 0 = \frac{E^2}{c^2} - |\vec{p}|^2 \qquad E = c|\vec{p}|$