Numbers + Sizes

Numbers give you some physical sense for the physics, and the interaction of the fermions via the bosons, along with a few other ideas like the uncertainty principal, give some characteristic lengths + energies.

Gravity

You all know that the acceleration of gravity at the earth's surface is $g \cong 9.8\ m/s^2$. Cavendish did one of the great experiments of all time by measuring (in his barn) the force between two masses, m_1 + m_2, due to gravity:

$$F_G = G_N \frac{m_1 m_2}{r^2} \quad \text{he got} \quad G_N \cong 6.67 \cdot 10^{-11} \frac{m^3}{kg\, s^2}$$

The cool thing is that he could then determine the mass of the earth from g and G_N!

$$\cancel{m} g = G_N \frac{m_E \cancel{m}}{R_E^2} \qquad R_E \simeq 6.38 \cdot 10^6\ m$$

$$m_E = \frac{9.8 \cdot (6.38 \cdot 10^6)^2}{6.67 \cdot 10^{-11}} = \underline{5.98 \cdot 10^{24}\ kg}$$

We know that most of the earth's mass consists of neutrons + protons, which each have a mass of:

$$m_n \simeq m_p \simeq 1.67 \cdot 10^{-27}\ kg$$

So the "baryon number" = Σ of neutrons + protons is about $N_B \simeq \frac{5.98\cdot 10^{24}}{1.67\cdot 10^{-27}} \simeq 3.51\cdot 10^{51}$

I want to drive at the point: what is the mean distance between these nucleons? The volume taken by each, on average, is:

$$V_B \simeq \frac{\frac{4\pi}{3} R_E^3}{N_B} \simeq L_B^3 \leftarrow L_B \text{ is the mean distance.}$$

$$L_B = \left(\frac{4\pi}{3N_B}\right)^{1/3} R_E \simeq 0.67\cdot 10^{-10}\ \text{m}$$

$$\simeq 0.67 \text{ Angstroms}$$

L_B is a "typical" atomic size. We can conclude that the matter in the earth is similar to the matter at the surface.

Let's go another step. What if the earth were compressed into a black hole? What would the mean spacing between then be?

Black Hole $\longrightarrow$ escape velocity $= c$

$$\text{or} \quad G_N \frac{m m_E}{R_{BH}} = \frac{1}{2} m c^2$$

$$R_{BH} \leq \frac{2 G_N m_E}{c^2}$$

$$\leq \frac{2\cdot 6.67\cdot 10^{-11}\cdot 5.98\cdot 10^{24}}{(3.00\cdot 10^{8})^2}$$

$$R_{BH} \leq 8.86\ \text{mm}$$

$$R_{BH} = (1.39\cdot 10^{-9}) R_E$$

This means that if you crushed the earth into a black hole, it would be smaller than $\simeq 1$ cm in size, about a billionth of the current size.

And, the spacing between these nucleons would shrink to:

$$L_{BH} \simeq (1.39 \cdot 10^{-9}) L_B$$
$$L_{BH} \simeq 9.4 \cdot 10^{-20} \text{ m}$$

By coincidence, these two lengths:

$$L_B \simeq 0.67 \text{ Angstroms} = 0.67 \cdot 10^{-10} \text{ m}$$
$$L_{BH} \simeq 9.4 \cdot 10^{-20} \text{ m}$$

cover the length scales of particle physics, which are:

10^{-10} m — atoms

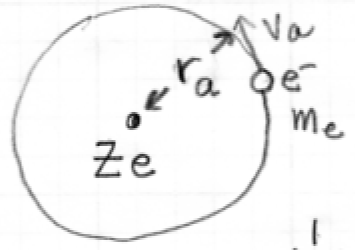
10^{-15} m — nucleons, baryons, mesons
(femtometer or fermi)

10^{-18} m — smallest distance probed in accelerators

Electromagnetism

+ m_e + uncertainty principle = size of atoms.

Use "Bohr Atom"



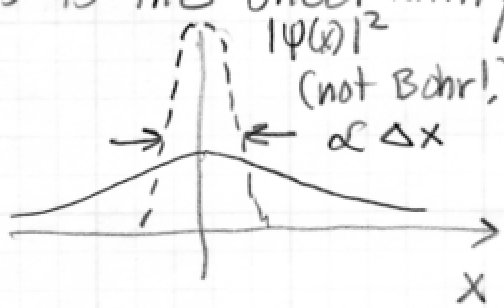
take nucleus ∞ heavy.

$$F = ma$$

$$k\frac{Ze^2}{r_a^2} = m_e\frac{v_a^2}{r_a} \qquad \leftarrow \text{2 unknowns, } v_a, r_a$$

$\frac{1}{4\pi\varepsilon_0}$ MKS

If there is no other physics, this situation would result in the "fall to the center" of the electron, giving atoms of infinitesimal size. What prevents this is the uncertainty principle.



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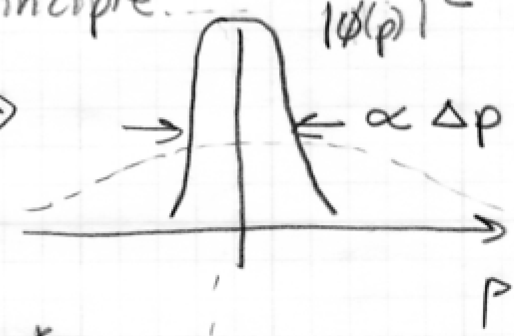
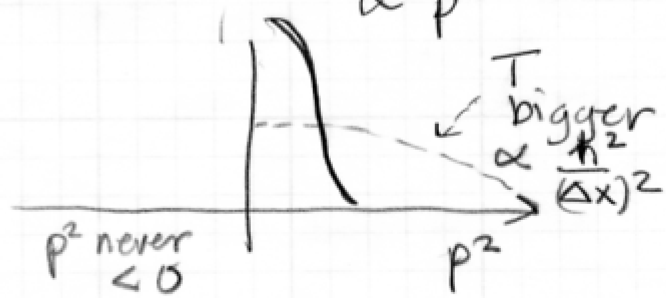


$$\Delta x \, \Delta p \geq \frac{\hbar}{2}$$

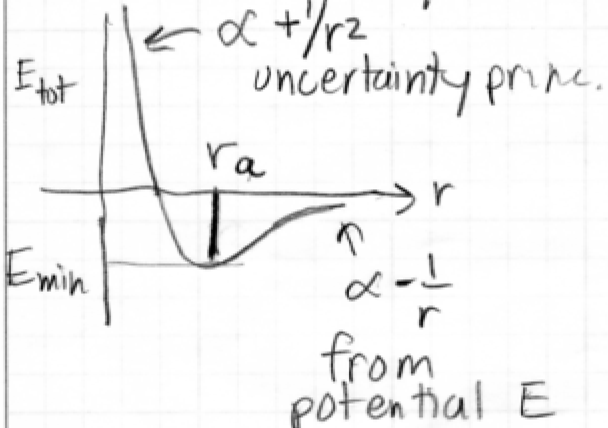
but T (kinetic E) $\propto p^2$

(sloppy, but good)



The point: as you squeeze electron to $\Delta x \sim r$ potential energy gets smaller as $1/r$ (coulomb), but mean kinetic energy blows up as $1/r^2$.

A compromise between the two that minimizes the total energy determines the size of atoms:



The semiclassical Bohr picture can be used to quantitatively get r_a.. the physics "feels" very different, however. The derivation involves generating a second equation in v + r:

$$m_e v_a r_a = n\hbar \qquad n = \text{integer}$$

→ First eliminate r_a $\qquad r_a = n\frac{\hbar}{m_e v_a}$

→ plug into F=ma ⇒ was $k\frac{Ze^2}{m_e} = v_a^2 r_a$

$$k\frac{Ze^2}{\cancel{m_e}} = v_a \cdot \frac{n\hbar}{\cancel{m_e}}$$

or, $\quad v_a = \frac{1}{n} \cdot \frac{kZe^2}{\hbar}$

better $\quad \beta_a = \frac{v_a}{c} = \frac{1}{n} k\frac{Ze^2}{\hbar c}$

Why better? Dimensionless → $k\frac{Ze^2}{\hbar c}$ dimensionless

and $k \cdot \frac{e^2}{\hbar c}$ involves only "fundamental" constants

$$\equiv \alpha \quad = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{\hbar c}\ (\text{MKS}),\ = \frac{e^2}{\hbar c}\ (\text{CGS}),\ = \frac{e^2}{4\pi\hbar c}\ (\text{HL})$$

$$\cong \frac{1}{137}$$

Then "speed" of electron $\simeq \beta_a \simeq \frac{Z}{n}\cdot\alpha$

Hydrogen $(Z=1)$, $n=1$, $\beta \simeq \alpha \simeq 1/137 \ll 1$
so H atoms are non-relativistic... relativity is a small effect for atoms up until $Z \sim 100$!

Size? $r_a = n \dfrac{\hbar}{m_e} \cdot \dfrac{1}{c \cdot \beta_a} = \dfrac{n^2}{Z} \dfrac{1}{\alpha} \dfrac{\hbar}{m_e c}$

$n=1,\ Z=1 \quad r_a \simeq 137 \cdot \dfrac{\hbar c}{m_e c^2}$

Interpretation: atoms are way bigger ($\times 137$) than the "natural" length scale associated with the electron, which is $\dfrac{\hbar}{m_e c} \equiv$ compton wavelength $= \dfrac{\hbar c}{m_e c^2}$ ← why?

$$\hbar c = 197.3 \text{ MeV} \cdot \underbrace{\text{fm}}_{10^{-15}\text{ m}}$$ (easy to remember $\simeq 200$).

"The conversion constant, converts length L in to energy $E = \dfrac{\hbar c}{L}$ and vice versa"!

then, $\dfrac{\hbar}{m_e c} = \dfrac{197.3 \text{ MeV} \cdot \text{fm}}{0.511 \text{ MeV}} \simeq 386 \text{ fm}$

"Bohr Radius" $= \dfrac{1}{\alpha} \cdot \dfrac{\hbar}{m_e c} = 52900 \text{ fm}$

$\simeq 0.529 \cdot 10^{-10} \text{ m} \simeq \frac{1}{2}$ Å

Atomic size scale.

note: $\alpha \cdot \dfrac{\hbar}{m_e c} = k \dfrac{e^2}{\cancel{\hbar} c} \cdot \dfrac{\cancel{\hbar}}{m_e c} = k \dfrac{e^2}{m_e c^2} \simeq 2.8 \text{ fm}$

is the "classical electron radius" — no $\hbar$

where E-field energy $\simeq m_e c^2$

What is E_{min}? $= -k\dfrac{Ze^2}{r_a} + \dfrac{1}{2}m_e v_a^2$

$$= -k\frac{Ze^2}{\underbrace{\frac{n^2}{Z}\frac{1}{\alpha}\frac{\hbar}{m_e c}}_{}} + \frac{1}{2}m_e\frac{Z^2}{n^2}\alpha^2 c^2$$

$$-1\, m_e \frac{Z^2}{n^2}\alpha^2 c^2$$

$$\boxed{E_{min} = -\frac{1}{2}\left(\frac{Z}{n}\right)^2 \alpha^2 m_e c^2}$$

In other words, when an electron is bound to a proton, the electron "loses" $\simeq \underbrace{-\frac{1}{2}\alpha^2}_{\simeq 2.7\cdot 10^{-5}}$ of its (# $n=1$) rest energy.

Since this is small, electrons bound in H atoms are, again, non-relativistic.

Strong Interaction

Recall that the electromagnetic interaction, for two unit charges of opposite sign separated by a distance r, gives a potential energy of

$$\tilde{U}(r) = -k\frac{e^2}{r} \qquad k = \frac{1}{4\pi\epsilon_0} \text{ MKS etc.}$$

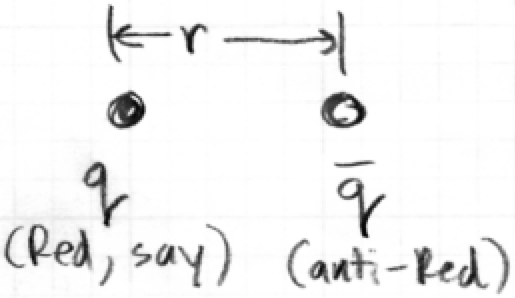
In particle physics we usually divide this expression by $\hbar c$

$$U(r) = \frac{\tilde{U}(r)}{\hbar c} = \frac{-ke^2}{\hbar c}\frac{1}{r} = -\frac{\alpha}{r}$$

so $U(r)$ has dimensions of length^{-1}, while $\tilde{U}(r)$ had dimensions of energy.

In this form, turns out the condition for electromagnetism to be a <u>perturbation</u> is that $\alpha \ll 1$; since $\alpha = \frac{1}{137}$, this condition is satisfied... and, $\alpha \ll 1$ keeps the problem non-relativistic

For the strong interaction, imagine a quark and an anti-quark



Prior to quantum corrections, the "chromo-dynamic" potential energy

$$\boxed{U_c(r) = -\frac{4}{3}\frac{\alpha_s}{r}}$$ (massless gluons)

Here, the $\frac{4}{3}$ comes from a little group theory of "Red-Green-Blue," and the "constant" α_s is like α for electromagnetism [$\propto$ product of charges]

"α_s" is pronounced "alpha-s".

The key distinction between electrodynamics and chromodynamics is:

$$\alpha_s \gg 1 \qquad (\alpha \ll 1)$$

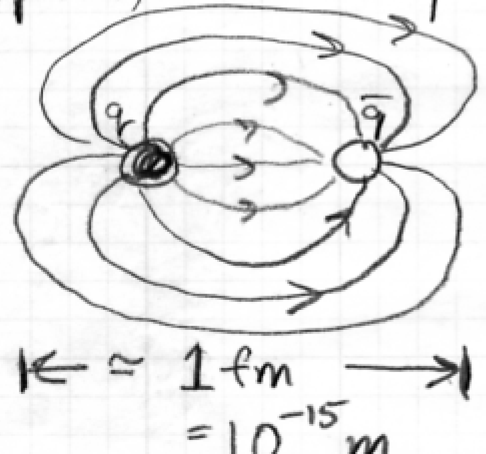
Consequences:

- strong force not a perturbation, usually
- quarks in bound states are sometimes highly relativistic
- α_s not a "constant", it depends on the process you're studying, because

The quantum corrections are <u>large</u>.

There is not a closed form solution to the problem of a $q-\bar{q}$ bound state, and even computer solutions are not so great, even now.

Qualitatively, the solution <u>has</u> been studied experimentally. The key point is that unlike the electromagnetic interaction, the color force is <u>confined</u> to a small region in space, as the quarks are:



consider $q = u$
$\bar{q} = \bar{d}$

Meson, charge +1

$\uparrow\downarrow$ ← spin 0, called a π^+

How does the size compare with $\frac{\hbar}{m_q c}$?

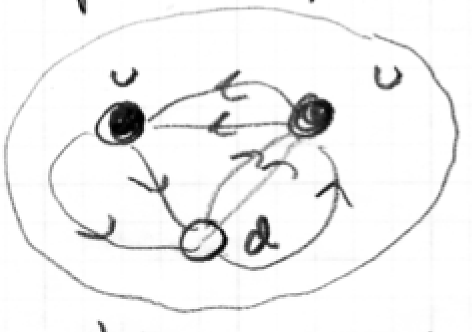
$$m_q c^2 \simeq \frac{1}{2}(3+6) \simeq 5\ \text{MeV}$$

$$\frac{\hbar c}{m_q c^2} \sim \frac{197.3\ \text{MeV-fm}}{5} \simeq 40\ \text{fm} \gg 1\ \text{fm}$$

<u>This</u> time, in <u>contrast</u> to the Hydrogen atom, the size of the bound state is way, way <u>smaller</u> than the compton wavelength of the constituent fermion. This is because the strong force is so strong that it smushes the wavefunctions down real small. Result... the momenta get really big! (uncertainty principle)

How big? $c\Delta p \sim \frac{\hbar c}{\Delta x} \sim 200$ MeV

Lets picture *your* protons



$\leftarrow M_p c^2 \simeq 938.3$ MeV

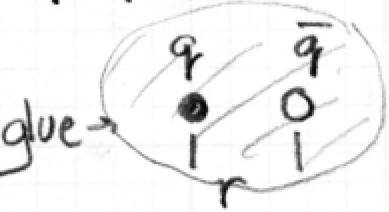
$2m_u c^2 + m_d c^2 \simeq 12$ MeV

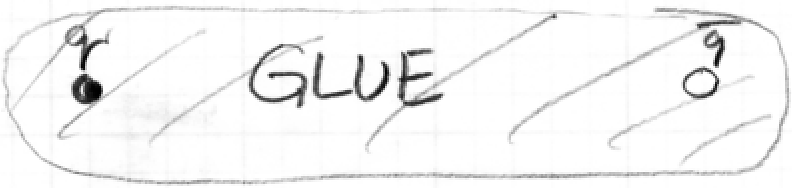
$|\leftarrow$ also $\simeq 1$ fm $\longrightarrow|$

most of the mass of the proton is *actually* "bottled up" kinetic energy of the quarks (!). A minority of the proton mass is actually from gluon field energy (10-30%). But only ~1% is from the mass of the quarks.

HOWEVER, sometimes the approximation that $m_q \simeq \frac{1}{3} M_p$ (for $q = u$ or d) is still useful. This is called the "constituent mass" of the quark. Only a few experiments are sensitive to the relativistic motion of the bound quarks, and show that the "current mass" $\simeq 5$ MeV is right.

Quantum corrections also change the q-$\bar{q}$ potential at *large* distances.





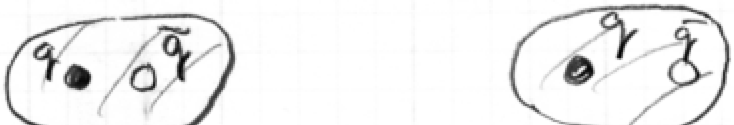
small:

$V(r) \propto \frac{1}{r}$

r large:

$V(r) \propto r$

energy so large, favorable to:



SNAP the GLUE