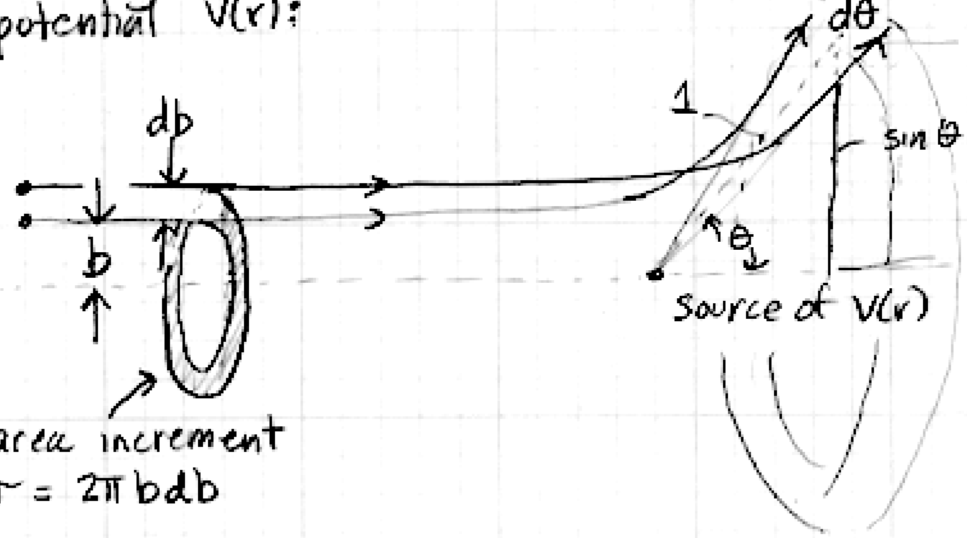
Differential Cross Sections

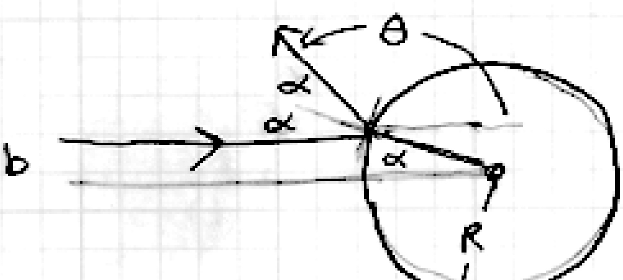
Imagine an infinitesimal particle scattering through a potential $V(r)$:



$d\Omega = 2\pi \sin\theta \, d\theta$

so $$\frac{d\sigma}{d\Omega} = \left| \frac{2\pi b}{2\pi \sin\theta} \frac{db}{d\theta} \right| = \left| \frac{b}{\sin\theta} \frac{db}{d\theta} \right|$$

Example: scattering off a hard sphere



$$2\alpha + \theta = \pi$$

$$b = R\sin\alpha$$

$$= R\sin\left[\frac{\pi}{2} - \frac{\theta}{2}\right]$$

$$b = R\cos(\theta/2)$$

$$\frac{db}{d\theta} = -\frac{R}{2}\sin\frac{\theta}{2}$$

so, $$\frac{d\sigma}{d\Omega} = \left| \frac{b}{\sin\theta} \times -\frac{R}{2}\sin\frac{\theta}{2} \right|$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{2}R^2 \frac{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\sin\theta} = \frac{1}{4}R^2}$$

$$\sigma = \int d\Omega \, \frac{1}{4}R^2 = \frac{1}{4}R^2 \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi = \frac{1}{4} \cdot R^2 \times 2 \times 2\pi$$

$$\boxed{\sigma = \pi R^2}$$

area in projection makes sense.

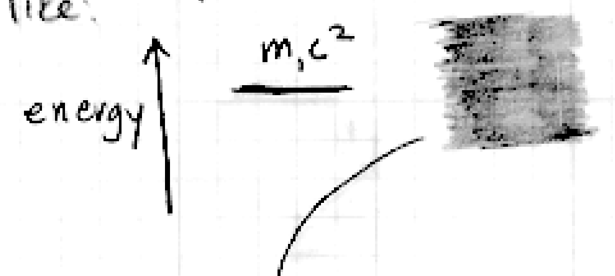
Golden Rule For Decays

The decay of a particle:

$$1 \rightarrow 2+3+4+\ldots+n$$

Has some strange properties. Must stem from the fact that in free space, the system $(2+3+4+\ldots n)$ has a continuum of energies available. The quantum mechanical computation would be way easier if the initial + final states were discrete (like the two state problems we've looked at).

The picture of the transition looks something like:



$\leftarrow \sum_{i=2}^{n} \sqrt{(c|\vec{p}_i|)^2 + (m_i c^2)^2}$

values of momentum

- The higher the density of states, the faster the transition
- The existence of the continuum means that one has to be really careful about doing the limiting cases. Crudely:

expect amplitude $\propto t$ (time)

prob. $\propto t^2$ (true if discrete, not continuum)

include continuum: prob. $\propto$ (density) $\times \Delta E \; t^2$ — eventually $\Delta E \to 0$

But, from uncertainty principal, $\Delta E t \sim$ constant

continuum $\underline{\text{prob} \propto t}$ alone.

or rate $\propto \frac{d(\text{prob})}{dt} \propto$ constant

specifically, rate $= \frac{2\pi}{\hbar} |\mathcal{M}|^2 \times$ (density of space)

$\mathcal{M} = \langle \text{final} | \underset{\sim}{\mathcal{H}} | \text{initial} \rangle$ $\quad \mathcal{H}$ = "perturbing" hamiltonian

The precise result (p. 195, 6.15) for the decay $1 \to 2+3+4+ \cdots n$ is:

$$d\Gamma = |\mathcal{M}|^2 \frac{S}{2\hbar m_1} \left[\left(\frac{c\, d^3 p_2}{(2\pi)^3 2E_2} \right) \cdot \left(\frac{c\, d^3 p_3}{(2\pi)^3 2E_3} \right) \cdots \left(\frac{c\, d^3 p_n}{(2\pi)^3 2E_n} \right) \right]$$

$$\times (2\pi)^4 \delta^4 (p_1 - p_2 - p_3 - \cdots - p_n)$$

where: $\mathcal{M}$ = transition matrix element, often estimated with Feynman diagrams

S = statistical factor for identical particles to make up for double counting; for each group of j identical particles, a multiplicative factor of $j!$ contributed

m_1 = mass of parent

$\vec{p}_2$ = 3-momentum of particle 2

$\vdots$

$\vec{p}_n$ = - - - - - - - - - n

$E_i = \sqrt{(m_i c^2)^2 + c^2 \vec{p}_i^{\,2}}$

p_i = 4 momentum $(E_i/c, \vec{p}_i)$

$\delta^4(p_1 - p_2 - \cdots p_n)$ = δ-function that insures $p_1 = p_2 + p_3 + \cdots$ for 4-momentum.

Usually, we want the <u>total</u> decay rate; to get that, integrate over all $\vec{p}_2, \vec{p}_3, \vec{p}_4, \cdots \vec{p}_n$

The most common decay is 2-body. Then

$$\Gamma = \frac{S}{\hbar m_1} \left(\frac{c}{4\pi} \right)^2 \frac{1}{2} \int \frac{|\mathcal{M}|^2}{E_2 E_3} \delta^4 (p_1 - p_2 - p_3) d^3 p_2\, d^3 p_3$$

the fun comes in dealing with the 0^{th} component of the $\delta^4(p_1 - p_2 - p_3)$. To ease into that, start with a decay to <u>massless</u> particles; then $\frac{E_2}{c} = |\vec{p}_2|$, $\frac{E_3}{c} = |\vec{p}_3|$

working in rest frame,

$$\vec{p}_1 = \vec{p}_2 + \vec{p}_3 = 0, \quad \text{and} \quad p_1 = \left(\frac{E_1}{c}, \vec{0}\right) = (m_1 c, \vec{0})$$

$$\delta^4(p_1 - p_2 - p_3) = \delta\left(m_1 c - \frac{E_2}{c} - \frac{E_3}{c}\right)\delta^3(\vec{0} - \vec{p}_2 - \vec{p}_3)$$

$$= \delta(m_1 c - |\vec{p}_2| - |\vec{p}_3|)\,\delta^3(-\vec{p}_2 - \vec{p}_3)$$

$$\Gamma = \frac{S}{\hbar m_1}\left(\frac{1}{4\pi}\right)^2 \frac{1}{2}\int \frac{|\mathcal{M}|^2}{|\vec{p}_2||\vec{p}_3|}\,\delta(m_1 c - |\vec{p}_2| - |\vec{p}_3|)\,\delta^3(-\vec{p}_2 - \vec{p}_3)\,d^3p_2\,d^3p_3$$

were $\frac{E_2}{c}, \frac{E_3}{c}$

do integral over $\vec{p}_3$

$$= \quad '' \quad \int \frac{|\mathcal{M}|^2}{|\vec{p}_2|^2}\,\delta(m_1 c - 2|\vec{p}_2|)\,d^3p_2$$

$d\phi\, d(\cos\theta)\, \vec{p}_2^{\,2}\, d|\vec{p}_2|$

indep of ϕ, θ

assume independent of θ, ϕ

$$= \frac{S}{\hbar m_1}\left(\frac{1}{4\pi}\right)^2 \frac{1}{2}\int \underbrace{|\mathcal{M}|^2 \times 4\pi \times \delta(m_1 c - 2|\vec{p}_2|)\,d|\vec{p}_2|}_{\text{can do, } d|\vec{p}_2| = \frac{1}{2}[2\,d|\vec{p}_2|]}$$

$$\boxed{\Gamma = \frac{S}{16\pi^2 \hbar m_1} \times |\mathcal{M}|^2}$$

General Case

can still do the integral over $\vec{p}_3$, but now the relationships for the 0^{th} components are more complex:

$$\frac{E_2}{c} = \sqrt{(m_2 c)^2 + |\vec{p}_2|^2} \qquad \frac{E_3}{c} = \sqrt{(m_3 c)^2 + |\vec{p}_3|^2}$$

m_2 was 0 before

integral over $\vec{p}_3$ makes $= -\vec{p}_2$

$$\Gamma = \frac{S}{\hbar m_1}\left(\frac{1}{4\pi}\right)^2 \frac{1}{2}\int \frac{|\mathcal{M}|^2}{\sqrt{(m_2 c)^2 + |\vec{p}_2|^2}\sqrt{(m_3 c)^2 + |\vec{p}_2|^2}}\,\delta\left(m_1 c - \sqrt{(m_2 c)^2 + |\vec{p}_2|^2} - \sqrt{(m_3 c)^2 + |\vec{p}_2|^2}\right) d$$

Looks hard! Really it isn't, though.

First, $|\vec{p}_2| = |\vec{p}^*|$ (before) $= \rho$ (Griffiths)

And, we even know the magnitude of the ρ that satisfies $m_1 c - \sqrt{(m_2c)^2+\rho_0^2} - \sqrt{(m_3c)^2+\rho_0^2} = 0$ ... we did this before:

$$\rho_0^2 = \frac{\overbrace{(m_1^4+m_2^4+m_3^4-2m_1^2m_2^2-2m_1^2m_3^2-2m_2^2m_3^2)}^{\lambda(m_1^2,m_2^2,m_3^2)}}{4m_1^2}c^2$$

Second, we are integrating over $d^3p_2 = d^3\rho$, but our δ-function is in terms of a FUNCTION of ρ, not ρ itself:

$$E(\rho) \equiv c\left[\sqrt{(m_2c)^2+\rho^2} + \sqrt{(m_3c)^2+\rho^2}\right]$$

$$dE = c\left[\frac{\frac{1}{2}\cdot 2\rho}{\sqrt{(m_2c)^2+\rho^2}} + \frac{\frac{1}{2}\cdot 2\rho}{\sqrt{(m_3c)^2+\rho^2}}\right]d\rho = c\left[\frac{\rho\sqrt{(m_3c)^2+\rho^2}+\rho\sqrt{(m_2c)^2+\rho^2}}{\sqrt{(m_2c)^2+\rho^2}\sqrt{(m_3c)^2+\rho^2}}\right]d\rho$$

$$dE = c^2\frac{\rho E}{E_2E_3}d\rho \qquad \text{or} \qquad d\rho = \frac{1}{c^2}\frac{E_2E_3}{E\rho}dE$$

note, when $\rho = 0$, $E = (m_2+m_3)c^2$

$$\Gamma = \frac{S}{\hbar m_1}\left(\frac{1}{4\pi}\right)^2\frac{1}{2}\int_0^\infty \frac{c^4|\mathcal{M}|^2}{E_2E_3}\underbrace{\delta\left(m_1c-\sqrt{(m_2c)^2+\rho^2}-\sqrt{(m_3c)^2+\rho^2}\right)}_{\delta(m_1c-\frac{E}{c})}\cdot 4\pi\,\rho^2\underset{\frac{1}{c^2}\frac{E_2E_3}{E\rho}}{d\rho}$$

$$\Gamma = \frac{S}{\hbar m_1}\left(\frac{1}{4\pi}\right)\frac{1}{2}\int_{(m_2+m_3)c^2}^\infty \frac{c^4|\mathcal{M}|^2}{E_2E_3}\delta\left(m_1c-\frac{E}{c}\right)\cdot\frac{E_2\cdot E_3}{c^2E\rho}\times\rho^2\,dE$$

$$\Gamma = \frac{S}{8\pi\hbar m_1}\int_{(m_2+m_3)c^2}^\infty |\mathcal{M}|^2\frac{\rho}{E}\delta\left(m_1c-\frac{E}{c}\right)dE \qquad = 0 \text{ if } (m_2+m_3) > m_1$$

when $m_2+m_3 < m_1$, δ-function gets $E = m_1c^2$, $\rho = \rho_0$, extra factor of c

$$\boxed{\Gamma = \frac{S}{8\pi\hbar m_1} \times \underset{\substack{\uparrow\\ \delta\text{-function}\\ \text{was in } E/c}}{c} \times \frac{\rho_0}{m_1c^2}\times|\mathcal{M}|^2 = \frac{S\times\rho_0}{8\pi\hbar m_1^2 c}|\mathcal{M}|^2}$$

Dimensions:

$[\Gamma] = 1/\text{time} \qquad [p_0] = \dfrac{\text{Energy}}{c} \qquad [\hbar] = \text{Energy}\cdot\text{time}$

$[m_1^2 c] = \left[m_1^2 c^4 \times \dfrac{1}{c^3}\right] = \dfrac{\text{Energy}^2}{c^3} \qquad [s] = \text{none}$

$\left[\dfrac{p_0}{\hbar m_1^2 c}\right] = \dfrac{E/c}{E\cdot\text{time}\cdot E^2/c^3} = \dfrac{1}{\text{time}} \cdot \dfrac{1}{E^2/c^2}$ ← dimensions of momentum^2

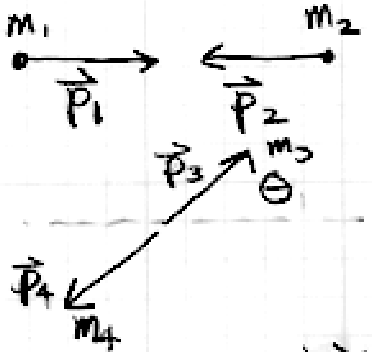
$\therefore\ [|\mathcal{M}|^2] = \text{momentum}^2; \quad [\mathcal{M}] = \text{momentum}$

Scattering $\quad 1+2 \to 3+4+5+\dots\ n$

$$d\sigma = |\mathcal{M}|^2 \times \frac{\hbar^2 S}{4\sqrt{(p_1\cdot p_2) - (m_1 m_2 c^2)^2}} \times \left[\left(\frac{c\,d^3p_3}{(2\pi)^3 2E_3}\right)\times\left(\frac{c\,d^3p_4}{(2\pi)^3 2E_4}\right)\times\cdots\left(\frac{c\,d^3p_n}{(2\pi)^3 2E_n}\right) \times (2\pi)^4\,\delta^4(p_1+p_2-p_3-p_4-p_5\cdots-p_n)\right]$$

Sloppy notation: $d\sigma$ is multiply differential, but we still just call it "$d\sigma$" before & after integrations over momenta

consider: initial / final



center of momentum frame.

$\vec{p}_1 = \vec{p}_i\ ,\ \vec{p}_2 = -\vec{p}_i$

$\vec{p}_3 = \vec{p}_f\ ,\ \vec{p}_4 = -\vec{p}_f$

when $m_1 \neq m_3\ ,\ m_2 \neq m_4$,

$$|\vec{p}_i|^2 = \frac{\lambda(\frac{s}{c^2}, m_1^2, m_2^2)}{4(s/c^2)}c^2, \quad |\vec{p}_f|^2 = \frac{\lambda(\frac{s}{c^2}, m_3^2, m_4^2)}{4(s/c^2)}c^2$$

Then $(p_1+p_2)^2 = m_1^2c^2 + m_2^2c^2 + 2p_1\cdot p_2$

or $p_1\cdot p_2 = \frac{1}{2}\left[s - m_1^2c^2 - m_2^2c^2\right]$

and
$$(p_1\cdot p_2)^2 - (m_1m_2c^2)^2 = \frac{1}{4}\left[s^2 + m_1^4c^4 + m_2^4c^4 - 2sm_1^2c^2 - 2sm_2^2c^2 + 2m_1^2m_2^2c^4\right] - (m_1m_2c^2)^2$$
$$= \frac{1}{4}c^4\lambda\left(\frac{s}{c^2}, m_1^2, m_2^2\right) = \frac{s}{c^2}\times c^4\cdot\frac{1}{c^2}|\vec{p}_i|^2$$

so $(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2 = s|\vec{p}_i|^2$

in the center of momentum frame: $s = \frac{1}{c}(E_1+E_2)^2$

so, $\sqrt{(p_1\cdot p_2)^2 - (m_1 m_2 c^2)^2} = \frac{1}{c}(E_1+E_2)|\vec{p}_i|$

$$d\sigma = |\mathcal{M}|^2 \times \frac{\hbar^2 S}{4\times\frac{1}{c}(E_1+E_2)|\vec{p}_i|} \times \frac{c\,d^3p_3}{(2\pi)^3 2E_3} \times \frac{c\,d^3p_4}{(2\pi)^3 2E_4} \times (2\pi)^4 \delta^4(p_1+p_2-p_3-p_4)$$

$$= \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2 c}{(E_1+E_2)|\vec{p}_i|} \times \frac{d^3p_3}{E_3}\frac{d^3p_4}{E_4}\delta^4(p_1+p_2-p_3-p_4)$$

now we integrate. the situation is very similar to the ~~decay~~ situation, except now:

1) we don't assume $|\mathcal{M}|^2$ is independent of θ. that leaves a $d\Omega$ integration undone.

2) $2 \to 3$ and $3 \to 4$.

$$\int \frac{d^3p_3\,d^3p_4}{E_3 E_4}\,\delta\left(\frac{1}{c}(E_1+E_2) - \sqrt{(m_3c)^2+|\vec{p}_3|^2} - \sqrt{(m_4c)^2+|\vec{p}_4|^2}\right)\delta^3(\underbrace{\vec{p}_1+\vec{p}_2}_{\substack{0 \\ \text{center OM}}}-\vec{p}_3-\vec{p}_4)$$

do this integral

$$= \int \frac{d\Omega\, p_3^2\,dp_3}{c^2\sqrt{(m_3c)^2+p_3^2}\times\sqrt{(m_4c)^2+p_3^2}}\,\delta\left(\frac{1}{c}(E_1+E_2) - \sqrt{(m_3c)^2+p_3^2} - \sqrt{(m_4c)^2+p_3^2}\right)$$

$$E(p_3) = c\left[\sqrt{(m_3c)^2+p_3^2} + \sqrt{(m_4c)^2+p_3^2}\right]$$

eventually: $dp_3 = \frac{1}{c^2}\frac{E_3E_4}{E\,p_3}\,dE$

$$= \int \frac{1}{c^2}\frac{d\Omega\, p_3\, dE}{E}\,\delta\left(\frac{1}{c}(E_1+E_2) - \frac{1}{c}E\right) = \frac{1}{c}\frac{|\vec{p}_f|}{E_1+E_2}\,d\Omega$$

so, $$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{(E_1+E_2)^2}\frac{|\vec{p}_f|}{|\vec{p}_i|}}$$

$[\mathcal{M}]$ = dimensionless