Differential Cross Sections

Imagine an infinitesimal particle scattering through a potential \( V(r) \):

\[
dr = 2\pi \sin \theta \, d\theta
\]

area increment \( d\sigma = 2\pi rb \, db \)

so

\[
\frac{d\sigma}{d\Omega} = \frac{2\pi b}{2\pi \sin \theta} \frac{db}{d\theta} = \frac{b}{\sin \theta} \frac{db}{d\theta}
\]

Example: scattering off a hard sphere

\[
2\pi + \alpha = \pi
\]

\[
b = b = R \sin \alpha = R \sin \left( \frac{\pi}{2} - \frac{\theta}{2} \right)
\]

\[
\frac{db}{d\theta} = -\frac{R}{2} \sin \frac{\theta}{2}
\]

\[
\frac{d\sigma}{d\Omega} = \frac{1}{2} R^2 \sin \theta \frac{\cos \frac{\theta}{2} \, d\theta}{\sin \theta} = \frac{1}{4} R^2
\]

\[
0^\circ = \int d\Omega \frac{1}{4} R^2 = \frac{1}{4} R^2 \int_0^{2\pi} \, d\theta \sin \theta \int_0^{\pi/2} \, d\theta = \frac{1}{4} R^2 \times 2 \times 2\pi
\]

\[
0^\circ = \pi R^2
\]

area in projection makes sense.
Golden Rule For Decays

The decay of a particle:

$$1 \rightarrow 2 + 3 + 4 + \ldots + n$$

Has some strange properties. Most stem from the fact that in free space, the system $(2+3+4+\ldots+n)$ has a continuum of energies available. The quantum mechanical computation would be way easier if the initial and final states were discrete (like the two state problems we've looked at).

The picture of the transition looks something like:

$$\frac{m_c^2}{E} \rightarrow \sum_{i=2}^{\infty} \frac{1}{(E_i - E)^2 + (m_c)^2}$$

The higher the density of states, the faster the transition.

- The existence of the continuum means that one has to be really careful about doing the limiting cases. Crudely:

  - Expect amplitude $\propto (\text{time})^{1/2}$
  - Prob. $\propto (\text{energy})^{1/2}$

- Include continuum: prob. $\propto (\text{density}) \times \Delta E + \Delta E \rightarrow 0$

  - But, from uncertainty principal, $\Delta E \propto \text{constant}$

  - Continuum prob $\propto \text{constant}$ alone.

  - Or rate $\propto \frac{d \langle \text{prob} \rangle}{dt} \propto \text{constant}$

Specifically, rate $= \frac{2\pi}{h} \langle M^2 \rangle \times (\text{density of space})$

$M = \langle \text{final} | \hat{H} | \text{initial} \rangle$ \hat{H} = “perturbing Hamiltonian”
The precise result \((p.195, 6.15)\) for the decay \(1 \rightarrow 2 + 3 + 4 + \cdots + n\) is:

\[
\Delta \Gamma = 1 \frac{m^2}{2 \hbar^2 m_1} \left[ \frac{\left( \frac{c d^3 p_2}{(2m)^2 E_2} \right)}{\left( \frac{c d^3 p_2}{(2m)^2 E_2} \right)} \frac{\left( \frac{c d^3 p_3}{(2m)^2 E_3} \right)}{\left( \frac{c d^3 p_3}{(2m)^2 E_3} \right)} \cdots \right] 
\]

\[\times (2\pi)^4 \delta^4(p_1 - p_2 - p_3 - \cdots - p_n)\]

where: \(m\) = transition matrix element, often estimated with Feynman diagrams

\(S\) = statistical factor for identical particles to make up for double counting; for each group of \(j\) identical particles, a multiplicative factor of \(j!\) contributed

\(m_1\) = mass of parent

\(\vec{p}_2\) = 3-momentum of particle 2

\(\vec{p}_n\) = \(n\)-momentum

\(E_1 = \frac{(m_1 c^2)^2 + \vec{p}_1^2}{2}\)

\(p_i = 4\) momentum \((E/\gamma, \vec{p}_i)\)

\(\delta^4(p_1 - p_2 - \cdots - p_n) = \delta\)-function that insures \(p_i = p_2 + p_3 + \cdots\) for 4-momentum.

Usually, we want the total decay rate \(\Gamma\); to get that, integrate over all \(\vec{p}_2, \vec{p}_3, \vec{p}_4, \cdots, \vec{p}_n\).

The most common decay is 2-body. Then

\[
\Gamma = \frac{1}{4 \pi} \frac{m^2}{2 \hbar^2 m_1} \int \frac{d^3 p_2}{E_2} \delta^4(p_1 - p_2 - p_3) d^3 p_2 d^3 p_3
\]

the fun comes in dealing with the 0th component of the \(\delta^4(p_1 - p_2 - p_3)\). To ease into that, start with a decay to massless particles; then \(E_2 = \frac{1}{2} |\vec{p}_2|, E_3 = \frac{1}{2} |\vec{p}_3|\).
working in rest frame, 
\[ \vec{p}_1 = \vec{p}_2 + \vec{p}_3 = 0 \], and \( p_1 = \left( \frac{E_1}{c}, 0, 0 \right) = (m_1c, 0) \)

\[ \delta^+(p_1 - p_2 - p_3) = \delta(p_1 - p_2 - p_3) \]

\[ = \delta(m_1c - \frac{E_1}{c} - E_2) \delta^3(0 - \vec{p}_2 - \vec{p}_3) \]

\[ = \delta(m_1c - \vec{p}_2 - \vec{p}_3) \]

\[ \Gamma = \frac{S}{\hbar m_1 \sqrt{2}} \left( \frac{1}{4\pi} \right)^{\frac{3}{2}} \int \frac{1}{(\vec{p}_2 \cdot \vec{p}_3)^2} \delta(m_1c - \vec{p}_2 - \vec{p}_3) \delta^3(0 - \vec{p}_2 - \vec{p}_3) d^3p_2 d^3p_3 \]

\[ \text{do integral over } \vec{p}_3 \]

\[ = \frac{S}{\hbar m_1 (4\pi)^{\frac{3}{2}}} \int \frac{1}{(\vec{p}_2)^2} \delta(m_1c - 2\vec{p}_2) d^3p_2 \]

\[ \text{assume } \delta \text{ independent of } \vec{p}_3 \]

\[ \text{can do } d^3\vec{p}_3 = \frac{1}{2} d\vec{p}_2 \]

\[ \Gamma = \frac{S}{16\pi^2 \hbar m_1} \]

**General Case**

can still do the integral over \( \vec{p}_3 \), but now the relationships for the \( \delta^0 \) components are more complex:

\[ \frac{E_2}{c} = \sqrt{(m_2c)^2 + \vec{p}_2^2} \]

\[ \text{integrate over } \vec{p}_2 \]

\[ \Gamma = \frac{S}{\hbar m_1 (4\pi)^{\frac{3}{2}}} \int \frac{1}{(m_2c)^2 + \vec{p}_2^2} \delta(m_1c - \vec{p}_2) d\vec{p}_2 \]

Looks hard! Really it isn't, though.
First, $|\beta_2| = |p^*| \text{ (before)} = p$ (Griffiths).

And, we even know the magnitude of the $p$ that satisfies $m_1 c - \sqrt{(m_2^2 + p_0^2) - (m_3^2 + p_0^2)} = 0$. We did this before:

$$p_0^2 = \frac{m_1^2 + m_2^4 + m_3^4 - 2m_1^2m_2^2 - 2m_2^2m_3^2}{4 m_1^2}.$$  

Second, we are integrating over $d^3 p_2 = d^3 p$, but our $\delta$-function is in terms of a functional of $p$, not $p$ itself:

$$E(p) = c \left[ \sqrt{(m_2^2 + p_3^2) + \sqrt{(m_3^2 + p_3^2)}} \right]$$

$$dE = c \left[ \frac{1}{p_3^2} - \frac{1}{v(m_2^2 + p_3^2)} - \frac{1}{v(m_3^2 + p_3^2)} \right] \frac{d^3 p}{E_2 E_3}$$

Note, when $p = 0$, $E = (m_2 + m_3) c^2$

$$\Pi = \frac{S}{\hbar m_1} \frac{(1)}{(4\pi)^{1/2}} \int \frac{c^4 |M|^2}{E_2 E_3} \delta(m_1 c - (m_2^2 + p_3^2) - (m_3^2 + p_3^2)) \frac{4 \pi p_3^2 d^3 p}{E_2 E_3}$$

$$\Pi = \frac{S}{\hbar m_1} \frac{(1)}{(4\pi)^{1/2}} \int \frac{c^4 |M|^2}{E_2 E_3} \delta(m_1 c - E/c) \frac{E_2 E_3 \times 4 \pi p_3^2 d^3 p}{c^2 E_p}$$

$$\Pi = \frac{S}{S\hbar m_1 (m_2 + m_3)^2} \int |M|^2 \delta(m_1 c - E/c) dE = 0 \text{ if } (m_2 + m_3) > m_1$$

When $m_2 + m_3 < m_1$, $\delta$-function gets

$$E = m_1 c^2, \quad p = p_0, \text{ extra factor of } c$$

$$\Pi = \frac{S}{\hbar m_1} \times \frac{c}{m_1 c^2} \times \frac{p_0}{m_1 c^2} \times |M|^2 = \frac{S \times p_0}{\hbar m_1} \frac{|M|^2}{m_1 c^2}$$

was in $E/c$
\[
\text{Dimensions:} \quad [T] = \text{time} \quad [\mathbf{P}] = \frac{\text{Energy}}{c} \quad [\hbar] = \text{Energy} \cdot \text{time} \\
[m^2c^2] = \left[ \frac{m^2c^4}{\hbar^2} \right] = \frac{\text{Energy}^2}{\hbar^2} \quad [S] = \text{none} \\
\left[ \frac{\mathbf{P} \cdot \mathbf{v}}{\hbar m^2c^2} \right] = \frac{E/c}{E \cdot \text{time} \cdot E/c^3} = \frac{1}{\text{time}} \cdot \frac{E^2c^2}{\hbar^2} \quad \text{dimensions of momentum} \\
\Rightarrow \quad [\mathbf{m}^2] = \text{momentum}^2 \quad [\mathbf{M}] = \text{momentum} \\
\text{Scattering:} \quad n+2 \rightarrow 3+4+5+\cdots+n
\]

\[
d\sigma = |\mathbf{M}|^2 \times \frac{4\pi^2S}{4V(p_1, p_2) - (m_1, m_2, c^2)^2} \times \left\{ \left( \frac{c dP_3}{dt} \right)^2 \right\} \times \left( \frac{c dP_4}{dt} \right)^2 \times \left( \frac{c dP_5}{dt} \right)^2 \times \cdots \times (2n)! \times \delta^4(p_1, p_2, p_3, p_4, \ldots, p_n)
\]

Sloppy notation: \( d\sigma \) is multiply differential, but we still just call it "\( d\sigma \)" before and after integrations over momenta.

consider: \( m_1 \) initial \( \mathbf{P}_1 \) \( \mathbf{P}_2 \) \( \mathbf{P}_3 \) \( \mathbf{P}_4 \) \( \mathbf{P}_5 \)

final \( m_2 \) \( \mathbf{P}_2 \) \( \mathbf{P}_3 \) \( \mathbf{P}_4 \) \( \mathbf{P}_5 \)

Center of momentum frame

Initial \( \mathbf{P}_1 = \mathbf{P}_1 \) \( \mathbf{P}_2 = -\mathbf{P}_2 \)

Final \( \mathbf{P}_3 = \mathbf{P}_3 \) \( \mathbf{P}_4 = -\mathbf{P}_4 \) \( \mathbf{P}_5 = -\mathbf{P}_5 \)

when \( m_1 \neq m_2, \quad m_2 \neq m_4 \)

\[
|\mathbf{P}_1|^2 = \frac{\lambda(\sqrt{m_1^4m_2^3m_4^3})}{4(s/c^2)} \quad |\mathbf{P}_1|^2 = \frac{\lambda(\sqrt{m_2^4m_3^3m_4^3})}{4(s/c^2)}
\]

Then \( (\mathbf{P}_1 + \mathbf{P}_2)^2 = m_1^2c^2 + m_2^2c^2 + 2\mathbf{P}_1 \cdot \mathbf{P}_2 \)

or \( \mathbf{P}_1 \cdot \mathbf{P}_2 = \frac{1}{2} \left[ s - m_1^2c^2 - m_2^2c^2 \right] \)

And \( (\mathbf{P}_1 \cdot \mathbf{P}_2)^2 = (m_1m_2c^2)^2 = \frac{1}{4} \left[ s^2 + m_1^4c^4 + m_2^4c^4 - 2s^2m_1^2 - 2s^2m_2^2 + 2m_1^2m_2^2 \right] \)

\[
- (m_1m_2c^2)^2 = \frac{1}{4} \lambda(\sqrt{m_1^4m_2^3m_4^3}) = \frac{s}{c^2} \times c^4 \frac{1}{c^2} \quad |\mathbf{P}_1|^2 \]

\[ (p_1 \cdot p_2)^2 - (m, m, c^2)^2 = \frac{-s}{c^2} |\vec{p}_1|^2 \]

In the center of momentum frame: 
\[ s = \frac{1}{c^2} (E_1 + E_2)^2 \]

so, 
\[ (p_1 \cdot p_2)^2 - (m, m, c^2)^2 = \frac{1}{c^2} (E_1 + E_2) |\vec{p}_1|^2 \]

\[ d\sigma = \frac{|M|^2}{4 \pi} \frac{\hbar S_1}{(E_1 + E_2)|\vec{p}_1|^2} \frac{d^3p_2}{E_3} \frac{d^3p_4}{E_4} \delta^4(p_1 + p_2 - p_3 - p_4) \]

\[ = \left( \frac{\hbar c}{8\pi} \right)^2 \frac{S_1 |M|^2 c}{(E_1 + E_2)|\vec{p}_1|^2} \frac{d^3p_2}{E_3} \frac{d^3p_4}{E_4} \delta^4(p_1 + p_2 - p_3 - p_4) \]

Now we integrate. The situation is very similar to the decay situation, except now:
1) We don't assume $|M|^2$ is independent of $\theta$ that leaves a $d\Omega$ integration undone.
2) $2 \to 3$ and $3 \to 4$.

\[ \int \frac{d^3p_2}{E_3} \frac{d^3p_3}{E_4} \delta^3(p_1 + p_2 - p_3 - p_4) \delta^4(p_1 + p_2 - p_3 - p_4) \]

\[ = \int \frac{d\Omega^p_2 dp_2}{(m_0 c^2 + p_3^2)^2 + (m_0 c^2 + p_3^2)} \delta \left( \frac{1}{c^2} (E_1 + E_2) - \frac{1}{c^2} (E_3 + E_4) \right) \]

\[ = \int \frac{1}{c^2} \frac{d\Omega^p_2 dp_2}{E} \delta \left( \frac{1}{c^2} (E_1 + E_2) - \frac{1}{c^2} E \right) \frac{dE}{c^2 E} \]

\[ = \frac{|\vec{p}_1|}{c} d\Omega \]

\[ \frac{d\sigma}{d\Omega} = \left( \frac{\hbar c}{8\pi} \right)^2 \frac{|M|^2}{(E_1 + E_2)^2} \frac{|\vec{p}_1|}{1^2} \]

$|M| = \text{dimensionless}$