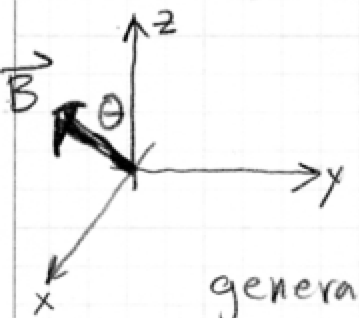
Diagonalizing an arbitrary Hermitian 2×2

Recall, eigenvectors of.. $B_x\hat{x} + B_z\hat{z} = B(\sin\theta\,\hat{x} + \cos\theta\,\hat{z})\cdot\vec{\sigma}$



are: $\underset{\sim}{U}(\theta\hat{y})\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix}$

$\begin{pmatrix}0\\1\end{pmatrix} = \qquad \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix}$

<u>Suppose</u> one wants to diagonalize the general $\boxed{\text{REAL VALUED}}$ 2×2 Hamiltonian

$\underset{\sim}{H} \doteq \begin{pmatrix}H_{11} & H_{12}\\ H_{21} & H_{22}\end{pmatrix}$ we are assuming this is real valued, then $H_{12} = H_{21} \equiv H_x$, since $H_x\underset{\sim}{\sigma}_x \doteq \begin{pmatrix}0 & H_x\\ H_x & 0\end{pmatrix}$

Then $\begin{pmatrix}H_{11} & 0\\ 0 & H_{22}\end{pmatrix} = \frac{1}{2}(H_{11}+H_{22})\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}\frac{1}{2}(H_{11}-H_{22}) & 0\\ 0 & -\frac{1}{2}(H_{11}-H_{22})\end{pmatrix}$

denote $\bar{H} \equiv \frac{1}{2}(H_{11}+H_{22}) \quad H_z \equiv \frac{1}{2}(H_{11}-H_{22}) \longrightarrow H_z\underset{\sim}{\sigma}_z$

so, the Hamiltonian $\underset{\sim}{H} \doteq \begin{pmatrix}H_{11} & H_{12}\\ H_{21} & H_{22}\end{pmatrix} \doteq \bar{H}\underset{\sim}{\mathbb{1}} + H_x\underset{\sim}{\sigma}_x + H_z\sigma_z$ -- Compare

we can "read off" eigenvectors, values.... $\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix}$ e.val. $\bar{H} + \sqrt{H_x^2+H_z^2}$

with $\cos\theta \equiv \dfrac{H_z}{\sqrt{H_x^2+H_z^2}}$

$\begin{pmatrix}-\sin\theta/2\\ \cos\theta/2\end{pmatrix}$ e.val $\bar{H} - \sqrt{H_x^2+H_z^2}$

($\bar{H}\underset{\sim}{\mathbb{1}}$ is always diagonal!)

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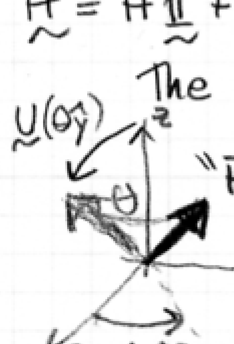
what about when H_{12} not purely real? $H_{12} = \mathrm{Re}(H_{12}) - i\,\mathrm{Im}(H_{12})$

In this case, $H_x \equiv \mathrm{Re}(H_{12})$ $H_y \equiv \mathrm{Im}(H_{12})$, $\bar{H}$ & H_z as before.

$\underset{\sim}{H} = \bar{H}\underset{\sim}{\mathbb{1}} + H_x\underset{\sim}{\sigma}_x + H_y\underset{\sim}{\sigma}_y + H_z\underset{\sim}{\sigma}_z$

The eigenvectors of This: $\underset{\sim}{U}(\phi\hat{z})\underset{\sim}{U}(\theta\hat{y})\begin{pmatrix}1\\0\end{pmatrix}$



$\begin{pmatrix}0\\1\end{pmatrix}$

$\begin{pmatrix}e^{-\frac{i\phi}{2}} & 0\\ 0 & e^{+\frac{i\phi}{2}}\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos\frac{\theta}{2}e^{-i\phi/2}\\ \sin\frac{\theta}{2}e^{i\phi/2}\end{pmatrix}$

$\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-\sin\frac{\theta}{2}e^{-i\phi/2}\\ \cos\frac{\theta}{2}e^{i\phi/2}\end{pmatrix}$

eigenvalues $\bar{H} \pm \sqrt{H_x^2+H_y^2+H_z^2} = \bar{H} \pm \left[|H_{12}|^2 + \frac{1}{4}(H_{11}-H_{22})^2\right]^{1/2}$

Isospin

the up and down quarks are very nearly identical... indeed, they can be thought of <u>as</u> identical, in the limits that:

1) electromagnetism's $\alpha \to 0$
2) the energy from gluon fields in mesons & baryons is $\gg m_u c^2$, $m_d c^2$

In the limit that they are really identical, then it would be very hard to agree about which is which. Indeed, one observer might call the u & d $\binom{u}{d}$, while another might say $\binom{u'}{d'}$, where

$$\begin{pmatrix} u' \\ d' \end{pmatrix} = \underset{\sim}{U}(\theta \hat{n}) \begin{pmatrix} u \\ d \end{pmatrix} \quad \leftarrow \text{like "spin" axes rotated in "flavor" space}$$

$\Rightarrow$ "SU(2) of <u>isospin</u>"

The most interesting consequences of this symmetry arise when one builds up complex systems out of up & down quarks. This is our first encounter with the "addition of angular momentum," although in this case, its not really angular momentum.

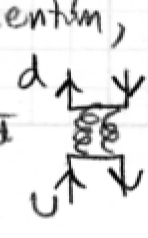
Idea is:

$$\left.\begin{array}{lll} u \text{ quark} & \text{is "isospin-up"}, & I_3 = +1/2 \\ \bar{u} & \text{" -down}, & I_3 = -1/2 \\ d \text{ quark} & \text{is "isospin-down"}, & I_3 = -1/2 \\ \bar{d} & \text{is "isospin-up"}, & I_3 = +1/2 \end{array}\right\} \text{all } I = \tfrac{1}{2}$$

When you build up hadrons, the <u>third</u> component of I is strictly additive. You don't know where I (total) will end up, however (at least not yet).

$u\bar{u}$	$u\bar{d}$	$\bar{u}d$	$d\bar{d}$	
0	+1	−1	0	$= I_3$

Physically, $u\bar{u}$ & $d\bar{d}$ are not eigenstates. Particularly if they have <u>real</u> spin = 0 (↓↑) and no total orbital angular momentum, they can <u>mix</u> through

$u\bar{u} \to d\bar{d}$ 

$d\bar{d} \to u\bar{u}$ 

→ same strength (exactly)

→ $u\bar{d}$ never → $\bar{u}d$ (charge)

If I project the arbitrary state $|\Psi\rangle$ onto $|u\bar{u}\rangle, |u\bar{d}\rangle, |\bar{u}d\rangle, |d\bar{d}\rangle$
Then the Schrödinger equation is: $a_1 = \langle u\bar{d}|\Psi\rangle$, $a_2 = \langle u\bar{u}|\Psi\rangle$, $a_3 = \langle d\bar{d}|\Psi\rangle$

$$i\hbar \frac{\partial}{\partial t}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} E & 0 & 0 & 0 \\ 0 & E+A & A & 0 \\ 0 & A & E+A & 0 \\ 0 & 0 & 0 & E \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

E = energy without the mixing
A = transition amplitude

well, we've seen this before....
eigenstates are:

$u\bar{d} = \pi^+ \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, e.val. E $\quad \bar{u}d = \pi^- \Rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, e.val. E

$\frac{1}{\sqrt{2}}[u\bar{u} - d\bar{d}] = \pi^0 \Rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$, e.val. E ; $\frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}) \rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, e.val E+2A

$\Rightarrow$ "triplet" with e.val. E $\Rightarrow$ like a "spin-1" multiplet
π^+, π^0, π^- $\rightarrow$ "isotriplet," $\simeq$ same mass $\Rightarrow I = 1$

$\Rightarrow$ "singlet" with e.val. E+2A $\rightarrow$ higher mass
$I = 0$ (like η, but η also has some $s\bar{s}$ inside)

Book: pages 116 – 122

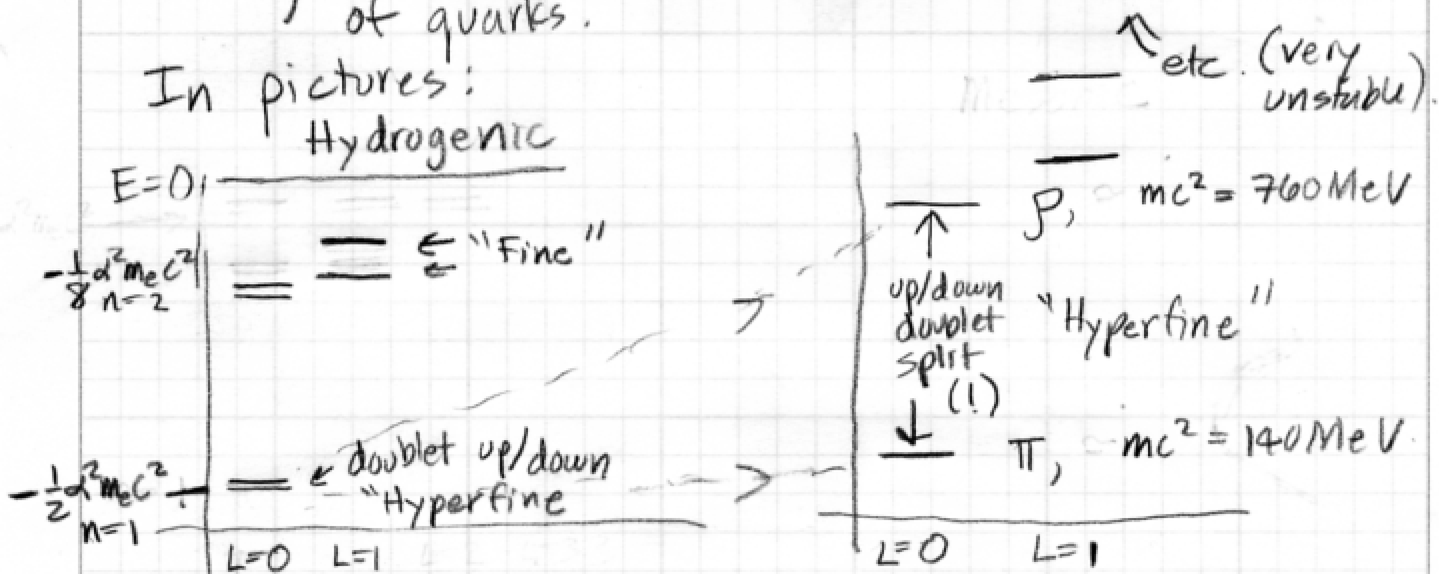
The Spin-Spin Interaction

The energy spectrum of bound states of quarks (hadrons) is qualitatively different than that of bound states of electrons + nucleii. However, the spectra are not unrelated; actually, the spectrum of hadrons is in many ways is simpler! This is because:

1) There is considerable degeneracy in atomic spectra: states with varying orbital angular momentum often have the same binding energy; the two states with electron spin up/down are also essentially degenerate. In quark systems, neither of these phenomenon are true. The orbital angular degeneracy is absent because the strong potential is not $1/r$; the spin degeneracy is broken because quarks have very large "color-magnetic moments.

2) There are fewer stable excited states of quarks.

In pictures:

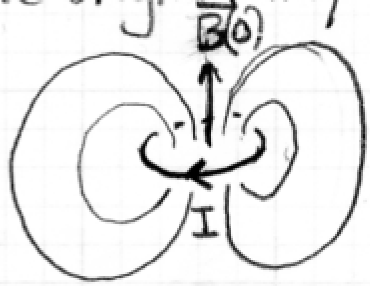


- Splitting of L=1 levels known as "Fine Structure," $\propto \alpha$ (e+m, Hydrogenic)
- Splitting of L=0 levels known as "Hyperfine" $\propto \alpha^2$ (e+m, Hydrogenic)
  - → minor effect in Hydrogen; nevertheless, the transition, through photon admission, allowed mapping of the Galaxy (21 cm line)
  - → Major effect in mesons.

Origin of Hyperfine Term

Text book sec 5.5, page 156

- Quarks, Leptons have "intrinsic" magnetic moments
- Like loops of current that have a vanishingly small radius
- Quarks – really a "color" magnetic moment
- There is a singularity in the magnetic field at the origin. Why?



$\rightarrow| 2a |\leftarrow$

$\mu = \pi a^2 I$ (moment)

$B(0) = \frac{\mu_0 I}{2a}$

$$\vec{B}_{far} = \frac{1}{r^3}\left[\frac{3(\vec{\mu}\cdot\vec{r})\vec{r}}{r^2} - \vec{\mu}\right] \times \frac{\mu_0}{4\pi}$$

$\downarrow$

imagine shrinking a, keeping μ constant so far field same!

$$\rightarrow \quad I = \frac{\mu}{\pi a^2}$$

$$B(0) = \frac{\mu \mu_0}{2\pi a^3} = \frac{\mu_0}{4\pi}\left[2 \cdot \frac{4\pi}{3} \frac{\mu}{\left(\frac{4\pi}{3}a^3\right)}\right] = \left(\frac{\mu_0}{4\pi}\right)\frac{8\pi}{3}\frac{\mu}{V}$$

- where $V = \frac{4\pi}{3}a^3$. $B(0)$ is singular as $1/V$
- if there is a non-zero probability of another particle sitting on the small moment, say with probability $|\psi(\vec{0})|^2$ (put small moment at 0), per unit volume, then that other particle will experience an average magnetic field of:

$$\langle \vec{B} \rangle = |\psi(\vec{0})|^2 \int_{\text{volume}} d^3x \left(\frac{\mu_0}{4\pi}\right)\frac{8\pi}{3}\frac{\vec{\mu}}{V}$$

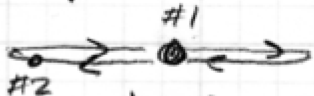
(volume has small moment inside)

$$= \left(\frac{\mu_0}{4\pi}\right)\frac{8\pi}{3}\vec{\mu}\,|\psi(\vec{0})|^2$$

another way of saying this: $\vec{B} = \vec{B}_{far} + \left(\frac{\mu_0}{4\pi}\right)\frac{8\pi}{3}\vec{\mu}\,\delta^3(\vec{r})$

infinitesimal dipole

This $\delta^3(\vec{r})$ arises in bound states for $L \neq 0$ states only. Pictures of orbits:



$L = 0$

sometimes #1 sits on #2

$\frac{\text{probability}}{\text{volume}} = |\psi(\vec{0})|^2$

0 separation

[consider further].

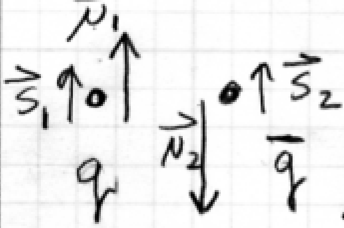
$L > 0$

never

(neglect) further

$$H = \langle -\vec{\mu}_2 \cdot \vec{B}_1 \rangle \quad \vec{B}_{far} \text{ averages to zero,}$$

$$= \frac{-\mu_0}{4\pi} \frac{8\pi}{3} \vec{\mu}_2 \cdot \vec{\mu}_1 |\psi(\vec{0})|^2$$



$$\vec{\mu}_2 = -\frac{g_2 q}{2m_2} \vec{s}$$

$$\mu_1 = \frac{g_1 q}{2m} \vec{s}_1 \qquad = -\frac{G}{m_2} \vec{s}_2$$

$$= \frac{G}{m_1} \vec{s}_1$$

− from antiparticle

here, q = "charge" could be <u>strong</u>.

$$H = +\left(\frac{\mu_0}{4\pi}\right) \cdot \left(\frac{8\pi}{3}\right) \cdot \frac{G^2}{m_1 m_2} \vec{s}_1 \cdot \vec{s}_2 |\psi(\vec{0})|^2$$

or

$$\boxed{H = +\frac{A}{m_1 m_2} \vec{s}_1 \cdot \vec{s}_2}$$

note: $\vec{s}_1 \downarrow \uparrow \vec{s}_2$ lower energy

$\vec{s}_1 \uparrow \uparrow \vec{s}_2$ higher energy.

Quantum Mechanical Version

$$\vec{\underset{\sim}{S}}_1 \cdot \vec{\underset{\sim}{S}}_2 \equiv \underset{\sim}{S}_{1x}\underset{\sim}{S}_{2x} + \underset{\sim}{S}_{1y}\underset{\sim}{S}_{2y} + \underset{\sim}{S}_{1z}\underset{\sim}{S}_{2z}$$

operates in #1's space   #2's space

A natural way to represent this operator is in the space of <u>products</u> of eigenstates of $\underset{\sim}{S}_{1z}$ + $\underset{\sim}{S}_{1z}$:

$\#1 = |\uparrow_1 \uparrow_2\rangle$ $\#2 = |\uparrow_1 \downarrow_2\rangle$ $\#3 = |\downarrow_1 \uparrow_2\rangle$ $\#4 = |\downarrow_1 \downarrow_2\rangle$

take one matrix element:

$$\langle \uparrow_1 \uparrow_2 | \vec{\underset{\sim}{S}}_1 \cdot \vec{\underset{\sim}{S}}_2 | \uparrow_1 \uparrow_2 \rangle$$

$$= \langle \uparrow_1 | \underset{\sim}{S}_{1x} | \uparrow_1 \rangle \langle \uparrow_2 | \underset{\sim}{S}_{2x} | \uparrow_2 \rangle + \langle \uparrow_1 | \underset{\sim}{S}_{1y} | \uparrow_1 \rangle \langle \uparrow_2 | \underset{\sim}{S}_{2y} | \uparrow_2 \rangle$$
$$+ \langle \uparrow_1 | \underset{\sim}{S}_{1z} | \uparrow_1 \rangle \langle \uparrow_2 | \underset{\sim}{S}_{2z} | \uparrow_2 \rangle$$

$$\langle \uparrow_1 | \underset{\sim}{S}_{1x} | \uparrow_1 \rangle = (1\ 0) \times \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}(1\ 0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$= \langle \uparrow_2 | \underset{\sim}{S}_{2x} | \uparrow \rangle = \langle \uparrow_1 | \underset{\sim}{S}_{1y} | \uparrow_1 \rangle = \langle \uparrow_2 | \underset{\sim}{S}_{2y} | \uparrow_2 \rangle$$

$$\langle \uparrow_1 | \underset{\sim}{S}_{1z} | \uparrow_1 \rangle = \frac{\hbar}{2} \qquad \langle \uparrow_2 | \underset{\sim}{S}_{2z} | \uparrow_2 \rangle = \frac{\hbar}{2}$$

so $\langle \uparrow_1 \uparrow_2 | \vec{\underset{\sim}{S}}_1 \cdot \vec{\underset{\sim}{S}}_2 | \uparrow_1 \uparrow_2 \rangle = +\frac{\hbar^2}{2}$

<u>also</u> $\langle \downarrow_1 \downarrow_2 | \vec{\underset{\sim}{S}}_1 \cdot \vec{\underset{\sim}{S}}_2 | \downarrow_1 \downarrow_2 \rangle = +\frac{\hbar^2}{2}$

claim: $\langle \uparrow_1 \uparrow_2 | \vec{\underset{\sim}{S}}_1 \cdot \vec{\underset{\sim}{S}}_2 | \text{other } 3 \rangle = 0$

$\uparrow_1 \downarrow_2,\ \downarrow_1 \uparrow_2,\ \downarrow_1 \downarrow_2$

first two: $\langle \uparrow_1 | \underset{\sim}{S}_{1x}, \underset{\sim}{S}_{1y} | \uparrow_1 \rangle = 0$ or $\langle \uparrow_2 | \underset{\sim}{S}_{2x}, \underset{\sim}{S}_{2y} | \uparrow_2 \rangle = 0$

and $\langle \uparrow_2 | \underset{\sim}{S}_{2z} | \downarrow_2 \rangle = 0$ and $\langle \uparrow_1 | \underset{\sim}{S}_{1z} | \downarrow \rangle = 0$

$$\langle \uparrow_1 \uparrow_2 | \vec{\underset{\sim}{S}}_1 \cdot \vec{\underset{\sim}{S}}_2 | \downarrow_1 \downarrow_2 \rangle = \langle \uparrow_1 | \underset{\sim}{S}_{1x} | \downarrow_1 \rangle \langle \uparrow_2 | \underset{\sim}{S}_{2x} | \downarrow \rangle + \langle \uparrow_1 | \underset{\sim}{S}_{1y} | \downarrow \rangle \langle \uparrow_2 | \underset{\sim}{S}_{2y} | \downarrow$$

$$\langle\uparrow|\underset{\sim}{S}_x|\downarrow\rangle = \frac{\hbar}{2}(1\ 0)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{\hbar}{2}(1\ 0)\begin{pmatrix}0\\1\end{pmatrix} = \frac{\hbar}{2}$$

$$\langle\uparrow|\underset{\sim}{S}_y|\downarrow\rangle = \frac{\hbar}{2}(1\ 0)\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{\hbar}{2}(1\ 0)\begin{pmatrix}-i\\0\end{pmatrix} = \frac{\hbar}{2}\cdot -i$$

so $\langle\uparrow_1\uparrow_2|\underset{\sim}{S}_1\underset{\sim}{S}_2|\downarrow_1\downarrow_2\rangle = \left(\frac{\hbar}{2}\right)^2(1^2+(-i)^2) = 0\,!$

---

similarly, $\langle\downarrow_1\downarrow_2|\underset{\sim}{S}_1\underset{\sim}{S}_2|$ other 3 $\rangle = 0$

$\uparrow_1\downarrow_2,\ \downarrow_1\uparrow_2,\ \uparrow_1\uparrow_2$

matrix:

	$\vert\uparrow_1\uparrow_2\rangle$	$\vert\uparrow_1\downarrow_2\rangle$	$\vert\downarrow_1\uparrow_2\rangle$	$\vert\downarrow_1\downarrow_2\rangle$
$\langle\uparrow_1\uparrow_2\vert$	$\hbar^2/4$	0	0	0
$\langle\uparrow_1\downarrow_2\vert$	0 hermitian			0 herm.
$\langle\downarrow_1\uparrow_2\vert$	0 herm.			0 hermitian
$\langle\downarrow_1\downarrow_2\vert$	0	0	0	$\hbar^2/4$

$$\langle\uparrow_1\downarrow_2|\vec{\underset{\sim}{S}}_1\cdot\vec{\underset{\sim}{S}}_2|\uparrow_1\downarrow_2\rangle = \langle\uparrow_1\downarrow_2|\underset{\sim}{S}_{1z}\cdot\underset{\sim}{S}_{2z}|\uparrow_1\downarrow_2\rangle = -\hbar^2/4$$

$$= \langle\downarrow_1\uparrow_2|\underset{\sim}{S}_{1z}\cdot\underset{\sim}{S}_{2z}|\downarrow_1\uparrow_2\rangle = -\hbar^2/4$$

($\underset{\sim}{S}_x$ + $\underset{\sim}{S}_y$ terms vanish)

<u>leaving</u>:

$$\langle\uparrow_1\downarrow_2|\vec{\underset{\sim}{S}}_1\cdot\vec{\underset{\sim}{S}}_2|\downarrow_1\uparrow_2\rangle = \langle\uparrow_1|\underset{\sim}{S}_{1x}|\downarrow_1\rangle\langle\downarrow_2|\underset{\sim}{S}_{2x}|\uparrow_2\rangle + \langle\uparrow_1|\underset{\sim}{S}_{1y}|\downarrow_1\rangle\langle\downarrow_2|\underset{\sim}{S}_{2y}|\uparrow_2\rangle$$

$$\langle\uparrow_1|\underset{\sim}{S}_{1x}|\downarrow_1\rangle = \frac{\hbar}{2}(1\ 0)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{\hbar}{2}$$

$$\langle\downarrow_2|\underset{\sim}{S}_{2x}|\uparrow_2\rangle = \frac{\hbar}{2}(0\ 1)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}$$

$$\langle\uparrow_1|\underset{\sim}{S}_{1y}|\downarrow_1\rangle = \frac{\hbar}{2}(1\ 0)\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{\hbar}{2}(-i)$$

$$\langle\downarrow_2|\underset{\sim}{S}_{2y}|\uparrow_2\rangle = \frac{\hbar}{2}(0\ 1)\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}(+i)$$

$$\langle \uparrow_1 \downarrow_2 | \underset{\sim}{\vec{S}}_1 \cdot \underset{\sim}{\vec{S}}_2 | \downarrow_1 \uparrow_2 \rangle = \frac{\hbar^2}{4} + \frac{\hbar^2}{4}(-i)(i) = 2 \times \frac{\hbar^2}{4}$$

by Hermitian-ness:

$$= \langle \downarrow_1 \uparrow_2 | \underset{\sim}{\vec{S}}_1 \cdot \underset{\sim}{\vec{S}}_2 | \uparrow_1 \downarrow_2 \rangle$$

so,

$$\underset{\sim}{\vec{S}}_1 \cdot \underset{\sim}{\vec{S}}_2 \doteq \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

pure $|\uparrow\downarrow\rangle$ or $|\downarrow\uparrow\rangle$ are not eigenstates!

eigenvectors of:

$$\begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} = -\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{identity, all eigenvectors}} + \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

eigenvectors	e.val
$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\longrightarrow$	2
$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\longrightarrow$	-2

or,

$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \doteq |\uparrow_1 \uparrow_2\rangle$ evalue $\frac{\hbar^2}{4}$; $\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \doteq \frac{1}{\sqrt{2}}\left[|\uparrow_1 \downarrow_2\rangle + |\downarrow_1 \uparrow_2\rangle\right]$ same eigenvalue $\frac{\hbar^2}{4}$

$-1+2 = 1$

and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \doteq |\downarrow_1 \downarrow_2\rangle$ same eigenvalue $\hbar^2/4$

TRIPLET, $\underset{\sim}{S} = \underset{\sim}{S}_1 + \underset{\sim}{S}_2$; $S = 1$

$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|\uparrow_1 \downarrow_2\rangle - |\downarrow_1 \uparrow_2\rangle\right)$ e value $\boxed{-3 \times \frac{\hbar^2}{4}}$

$-1-2$

SINGLET, $S = 0$

both $\uparrow\downarrow$ states get -1 from "classical" dot product ± 2 from "transitions"