THE FORCE ON A DIPOLE IN AN EXTERNAL FIELD

11.4 Consider a small circular current loop of radius r, placed in the magnetic field of some other current system, such as a solenoid. In Fig. 11.9, a field **B** is drawn that is generally in the z direction. It is not a uniform field. Instead, it gets weaker as we proceed in the z direction; that is evident from the fanning out of the field lines. Let us assume, for simplicity, that the field is symmetric about the z axis. Then it resembles the field near the upper end of the solenoid in Fig. 11.1. The field represented in Fig. 11.9 does *not* include the magnetic field of the current ring itself. We want to find the force on the current ring caused by the other field, which we shall call, for want of a better name, the *external field*. The net force on the current ring due to its *own* field is certainly zero, so we are free to ignore its own field in this discussion.

If you study the situation in Fig. 11.9, you will soon conclude that there is a net force on the current ring. It arises because the external field **B** has an *outward* component B_r everywhere around the ring. Therefore if the current flows in the direction indicated, each element of the loop, dl, must be experiencing a downward force of magnitude $IB_r\,dl/c$. If B_r has the same magnitude at all points on the ring, as it must in the symmetrically spreading field assumed, the total downward force will have the magnitude

$$F = \frac{2\pi r I B_r}{c} \tag{16}$$

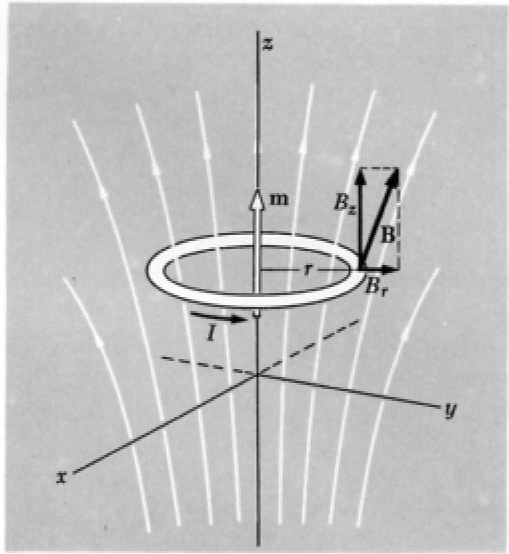


FIGURE 11.9
A current ring in an inhomogeneous magnetic field. (The field of the ring itself is not shown.) Because of the radial component of field B_r, there is a force on the ring as a whole.

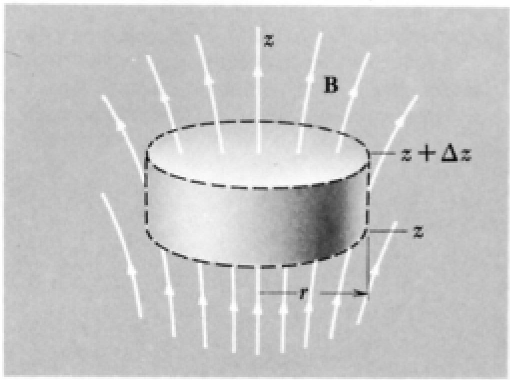
Now B_r can be directly related to the gradient of B_z. Since div **B** = 0 at all points, the net flux of magnetic field out of any volume is zero. Consider the little cylinder of radius r and height Δz (Fig. 11.10). The outward flux from the side is $2\pi r(\Delta z)B_r$ and the net outward flux from the end surfaces is

$$\pi r^2[-B_z(z) + B_z(z + \Delta z)]$$

which to the first order in the small distance Δz is $\pi r^2(\partial B_z/\partial z)\,\Delta z$. Setting the total flux equal to zero: $0 = \pi r^2(\partial B_z/\partial z)\,\Delta z + 2\pi r B_r\,\Delta z$,

$$B_r = -\frac{r}{2}\frac{\partial B_z}{\partial z} \tag{17}$$

FIGURE 11.10
Gauss' theorem can be used to relate B_r and $\partial B_z/\partial z$, leading to Eq. 17.



As a check on the sign, notice that, according to Eq. 17, B_r is positive when B_z is decreasing upward; a glance at the figure shows that to be correct.

The force on the dipole can now be expressed in terms of the gradient of the component B_z of the external field:

$$F = \frac{2\pi r I}{c}\,\frac{r}{2}\,\frac{\partial B_z}{\partial z} = \frac{\pi r^2 I}{c}\,\frac{\partial B_z}{\partial z} \tag{18}$$

In the factor $\pi r^2 I/c$ we recognize the magnitude m of the magnetic dipole moment of our current ring. So the force on the ring can be expressed very simply in terms of the dipole moment:

$$F = m \frac{\partial B_z}{\partial z} \tag{19}$$

We haven't proved it, but you will not be surprised to hear that for small loops of any other shape the force depends only on the *current* × *area* product, that is, on the dipole moment. The shape doesn't matter. Of course, we are discussing only loops small enough so that only the first-order variation of the external field, over the span of the loop, is significant.

Our ring in Fig. 11.9 has a magnetic dipole moment **m** pointing upward, and the force on it is downward. Obviously, if we could reverse the current in the ring, thereby reversing **m**, the force would reverse its direction. The situation can be summarized this way:

Dipole moment *parallel* to external field: Force acts in direction of *increasing* field strength.

Dipole moment *antiparallel* to external field: Force acts in direction of *decreasing* field strength.

Uniform external field: *Zero* force.

Quite obviously, this is not the most general situation. The moment **m** could be pointing at some odd angle with respect to the field **B**, and the different components of **B** could be varying, spatially, in different ways. It is not hard to develop a formula for the force **F** that is experienced in the general case. It would be exactly like the general formula we gave, as Eq. 10.23, for the force on an electric dipole in a nonuniform electric field. That is, the x component of force on any magnetic dipole **m** is given by

$$F_x = \mathbf{m} \cdot \text{grad } B_x \tag{20}$$

with corresponding formulas for F_y and F_z.