exceedingly rare. Of course, the proven existence of even one magnetically charged particle would have profound implications, but it would not alter the fact that in matter as we know, it the only sources of the magnetic field are electric currents. As far as we know,

$$\boxed{\operatorname{div} \mathbf{B} = 0 \qquad (everywhere)} \tag{1}$$

This takes us back to the hypothesis of Ampère, his idea that magnetism in matter is to be accounted for by a multitude of tiny rings of electric current distributed through the substance. We'll begin by studying the magnetic field created by a single current loop at points relatively far from the loop.

THE FIELD OF A CURRENT LOOP

11.3 A closed conducting loop, not necessarily circular, lies in the xy plane encircling the origin, as in Fig. 11.4*a*. A steady current I, measured in esu/sec, flows around the loop. We are interested in the magnetic field this current creates—not near the loop, but at distant points like P_1 in the figure. We shall assume that r_1, the distance to P_1, is much larger than any dimension of the loop. To simplify the diagram we have located P_1 in the yz plane; it will turn out that this is no restriction. This is a good place to use the vector potential. We shall compute first the vector potential $\mathbf{A}$ at the location P_1, that is, $\mathbf{A}(0, y_1, z_1)$. From this it will be obvious what the vector potential is at any other point (x, y, z) far from the loop. Then by taking the curl of $\mathbf{A}$ we shall get the magnetic field $\mathbf{B}$.

For a current confined to a wire, we had, as Eq. 35 of Chapter 6:

$$\underline{\text{cgs}} \qquad \underline{\text{MKS}}$$

$$\mathbf{A}(0, y_1, z_1) = \frac{I}{c}\int_{\text{loop}} \frac{d\mathbf{l}_2}{r_{12}} \quad \times \frac{\mu_0 c}{4\pi} \tag{2}$$

At that time we were concerned only with the contribution of a small segment of the circuit; now we have to integrate around the entire loop. Consider the variation in the denominator r_{12} as we go around the loop. If P_1 is far away, the first-order variation in r_{12} depends only on the coordinate y_2 of the segment $d\mathbf{l}_2$, and not on x_2. This should be clear from the side view in Fig. 11.4*b*. Thus, neglecting quantities proportional to $(x_2/r_{12})^2$, we may treat r_{12} and r'_{12}, which lie on top of one another in the side view, as equal. And in general, to first order in the ratio (loop dimension/distance to P_1), we have

$$r_{12} \approx r_1 - y_2 \sin\theta \tag{3}$$

Look now at the two elements of the path $d\mathbf{l}_2$ and $d\mathbf{l}'_2$ shown in

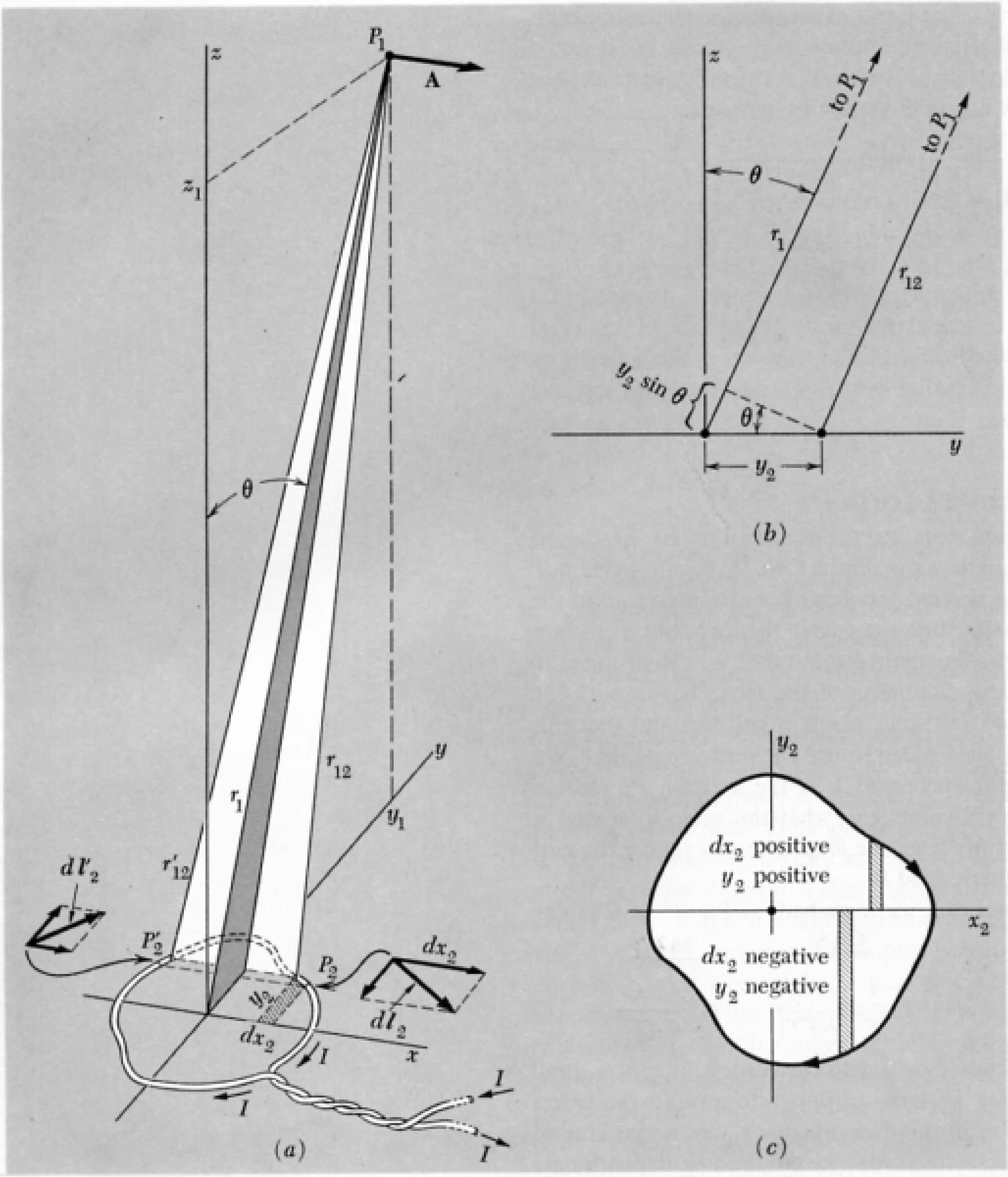


FIGURE 11.4
(*a*) Calculation of the vector potential **A** at a point far from the current loop. (*b*) Side view, looking in along the x axis, showing that

$$r_{12} \approx r_1 - y_2 \sin\theta \qquad \text{if } r_1 \gg y_2$$

(*c*) Top view, to show that $\int_{\text{loop}} y_2\, dx_2$ is the area of the loop.

Fig. 11.4*a*. For these the dy_2's are equal and opposite, and as we have already pointed out, the r_{12}'s are equal to first order. To this order then, their contributions to the line integral will cancel, and this will be true for the whole loop. Hence **A** at P_1 will not have a y component. Obviously it will not have a z component, for the current path itself has nowhere a z component. The x component of the vector potential comes from the dx part of the path integral. Thus

$$\mathbf{A}(0, y_1, z_1) = \hat{\mathbf{x}} \underbrace{\frac{I}{c}}_{\text{cgs}} \int \frac{dx_2}{r_{12}} \qquad \overset{\text{MKS}}{\times \frac{\mu_0 c}{4\pi}} \tag{4}$$

Without spoiling our first-order approximation, we can turn Eq. 3 into

$$\frac{1}{r_{12}} \approx \frac{1}{r_1}\left(1 + \frac{y_2 \sin\theta}{r_1}\right) \tag{5}$$

and using this for the integrand we have

$$\mathbf{A}(0, y_1, z_1) = \hat{\mathbf{x}} \frac{I}{cr_1} \int \left(1 + \frac{y_2 \sin\theta}{r_1}\right) dx_2 \quad \times \frac{\mu_0 c}{4\pi} \tag{6}$$

In the integration r_1 and θ are constants. Obviously $\int dx_2$ around the loop vanishes. Now $\int y_2\, dx_2$ around the loop is just the area of the loop, regardless of its shape (see Fig. 11.4c). So we get finally

$$\mathbf{A}(0, y_1, z_1) = \hat{\mathbf{x}} \frac{I \sin\theta}{cr_1^2} \times (\text{area of loop}) \quad \times \frac{\mu_0 c}{4\pi} \tag{7}$$

Here is a simple but crucial point: Since the *shape* of the loop hasn't mattered, our restriction on P_1 to the yz plane can't make any essential difference. Therefore we must have in Eq. 7 the general result we seek, if only we *state* it generally: The vector potential of a current loop of any shape, at a distance r from the loop which is much greater than the size of the loop, is a vector perpendicular to the plane containing $\mathbf{r}$ and the normal to the plane of the loop, of magnitude

$$A = \frac{Ia \sin\theta}{cr^2} \qquad \times \frac{\mu_0 c}{4\pi} \tag{8}$$

where a stands for the area of the loop.

This vector potential is symmetrical around the axis of the loop, which implies that the field $\mathbf{B}$ will be symmetrical also. The explanation is that we are considering regions so far from the loop that the details of the shape of the loop have negligible influence. All loops with the same *current* × *area* product produce the same far field. We call the product Ia/c the *magnetic dipole moment* of the current loop, and denote it by $\mathbf{m}$. The magnetic dipole moment is evidently a vector, its direction being that of the normal to the loop, or that of the vector $\mathbf{a}$, the directed area of the path surrounded by the loop

$$\mathbf{m} = \frac{I}{c}\mathbf{a} \qquad \times\ c \tag{9}$$

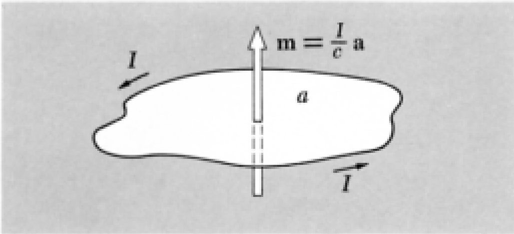


FIGURE 11.5
By definition, the magnetic moment vector is related to the current by a right-hand-screw rule as here shown.

As for sign, let us agree that the direction of $\mathbf{m}$ and the sense of positive current flow in the loop are to be related by a right-hand-screw rule, illustrated in Fig. 11.5. (The dipole moment of the loop in Fig. 11.4*a* points downward, according to this rule.) The vector potential for the field of a magnetic dipole $\mathbf{m}$ can now be written neatly with vectors:

cgs MKS

$$\mathbf{A} = \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2} \quad \times \frac{\mu_0}{4\pi} \tag{10}$$

where $\hat{\mathbf{r}}$ is a unit vector in the direction *from* the loop *to* the point for which $\mathbf{A}$ is being computed. You can check that this agrees with our convention about sign. Note that the direction of $\mathbf{A}$ must always be that of the current in the *nearest* part of the loop.

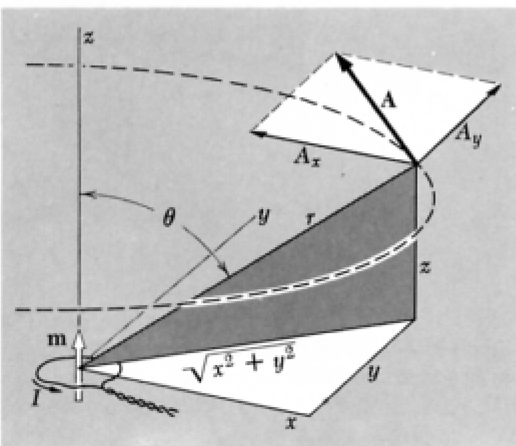
Figure 11.6 shows a magnetic dipole located at the origin, with the dipole moment vector $\mathbf{m}$ pointed in the positive z direction. To express the vector potential at any point (x, y, z), we observe that $r^2 = x^2 + y^2 + z^2$, and $\sin\theta = \sqrt{x^2 + y^2}/r$. The magnitude A of the vector potential at that point is

$$A = \frac{m \sin\theta}{r^2} = \frac{m\sqrt{x^2 + y^2}}{r^3} \quad \times \frac{\mu_0}{4\pi} \tag{11}$$

Since $\mathbf{A}$ is tangent to a horizontal circle around the z axis, its components are

$$\begin{aligned} A_x &= A\left(\frac{-y}{\sqrt{x^2 + y^2}}\right) = \frac{-my}{r^3} \quad \times \frac{\mu_0}{4\pi} \\ A_y &= A\left(\frac{x}{\sqrt{x^2 + y^2}}\right) = \frac{mx}{r^3} \quad \times \frac{\mu_0}{4\pi} \\ A_z &= 0 \end{aligned} \tag{12}$$

FIGURE 11.6
A magnetic dipole located at the origin. At every point far from the loop, *A* is a vector parallel to the *xy* plane, tangent to a circle around the *z* axis.



Let's evaluate $\mathbf{B}$ for a point in the xz plane, by finding the components of curl $\mathbf{A}$ and then (not before!) setting $y = 0$.

$$B_x = (\nabla \times \mathbf{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = -\frac{\partial}{\partial z}\frac{mx}{(x^2 + y^2 + z^2)^{3/2}} = \frac{3mxz}{r^5} \quad \cdot \frac{\mu_0}{4\pi}$$

$$B_y = (\nabla \times \mathbf{A})_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = \frac{\partial}{\partial z}\frac{-my}{(x^2 + y^2 + z^2)^{3/2}} = \frac{3myz}{r^5} \quad \cdot \frac{\mu_0}{4\pi}$$

$$\begin{aligned} B_z &= (\nabla \times \mathbf{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \\ &= m\left[\frac{-2x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^{5/2}} + \frac{x^2 - 2y^2 + z^2}{(x^2 + y^2 + z^2)^{5/2}}\right] = \frac{m(3z^2 - r^2)}{r^5} \quad \cdot \frac{\mu_0}{4\pi} \end{aligned} \tag{13}$$

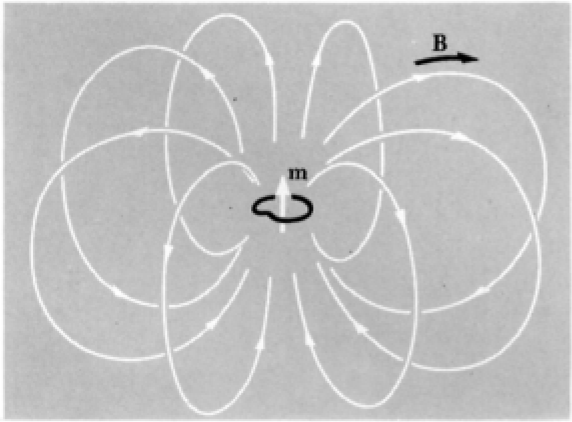


FIGURE 11.7
Some magnetic field lines in the field of a magnetic dipole, that is, a small loop of current.

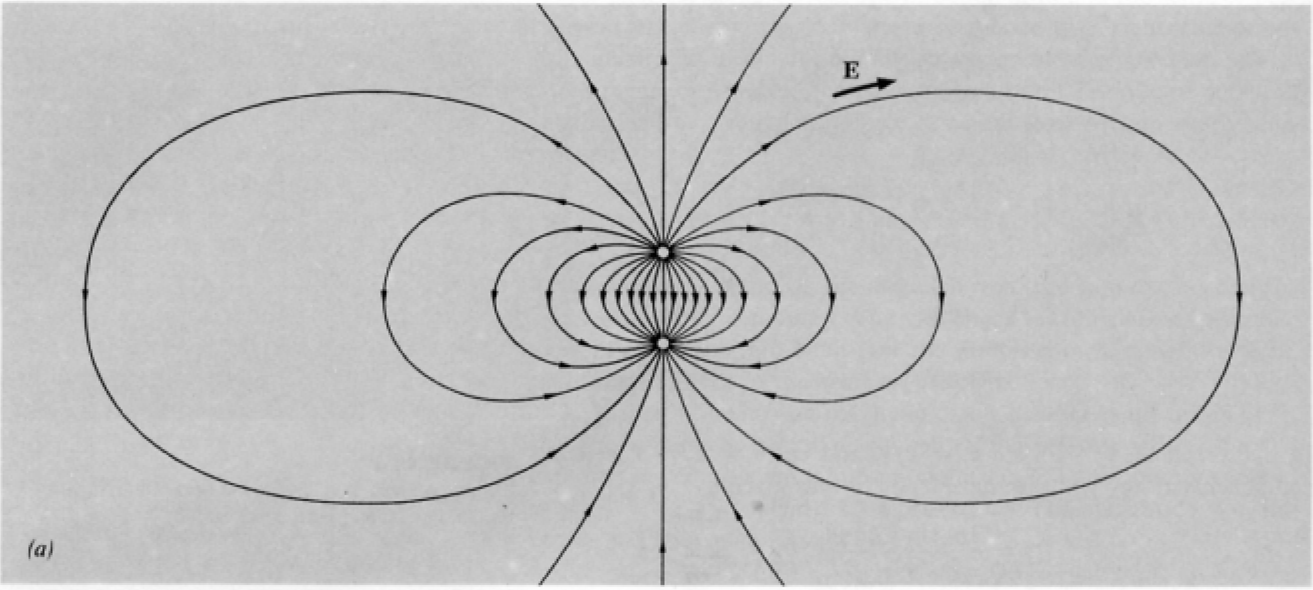
In the xz plane, $y = 0$, $\sin\theta = x/r$, and $\cos\theta = z/r$. The field components at any point in that plane are thus given by:

$$\begin{aligned} &\qquad\quad \text{cgs} \qquad\qquad\qquad \text{MKS} \\ B_x &= \frac{3m\sin\theta\cos\theta}{r^3} \quad \times \quad \frac{\mu_0}{4\pi} \\ B_y &= 0 \qquad\qquad\qquad\qquad\qquad (14) \\ B_z &= \frac{m(3\cos^2\theta - 1)}{r^3} \quad \times \quad \frac{\mu_0}{4\pi} \end{aligned}$$

Now turn back to Section 10.3, where in Eq. 10.14 we expressed the components in the xz plane of the field $\mathbf{E}$ of an electric dipole $\mathbf{p}$, which was situated exactly like our magnetic dipole $\mathbf{m}$. The expressions are identical. We have thus found that the magnetic field of a small current loop has at remote points the same form as the electric field of two separated charges. We already know what that field, the electric dipole field, looks like. Figure 11.7 is an attempt to suggest the three-dimensional form of the magnetic field $\mathbf{B}$ arising from our current loop with dipole moment $\mathbf{m}$. As in the case of the electric dipole, the field is described somewhat more simply in spherical polar coordinates:

$$B_r = \frac{2m}{r^3}\cos\theta \ {}^{\times \frac{\mu_0}{4\pi}} \qquad B_\theta = \frac{m}{r^3}\sin\theta \ {}^{\times \frac{\mu_0}{4\pi}} \qquad B_\varphi = 0 \qquad (15)$$

The magnetic field *close* to a current loop is entirely different from the electric field close to a pair of separated positive and negative charges, as the comparison in Fig. 11.8 shows. Notice that between the charges the electric field points down, while inside the current ring the magnetic field points up, although the far fields are alike. This reflects the fact that our magnetic field satisfies $\nabla \cdot \mathbf{B} = 0$ everywhere, *even inside the source*. The magnetic field lines don't end. By



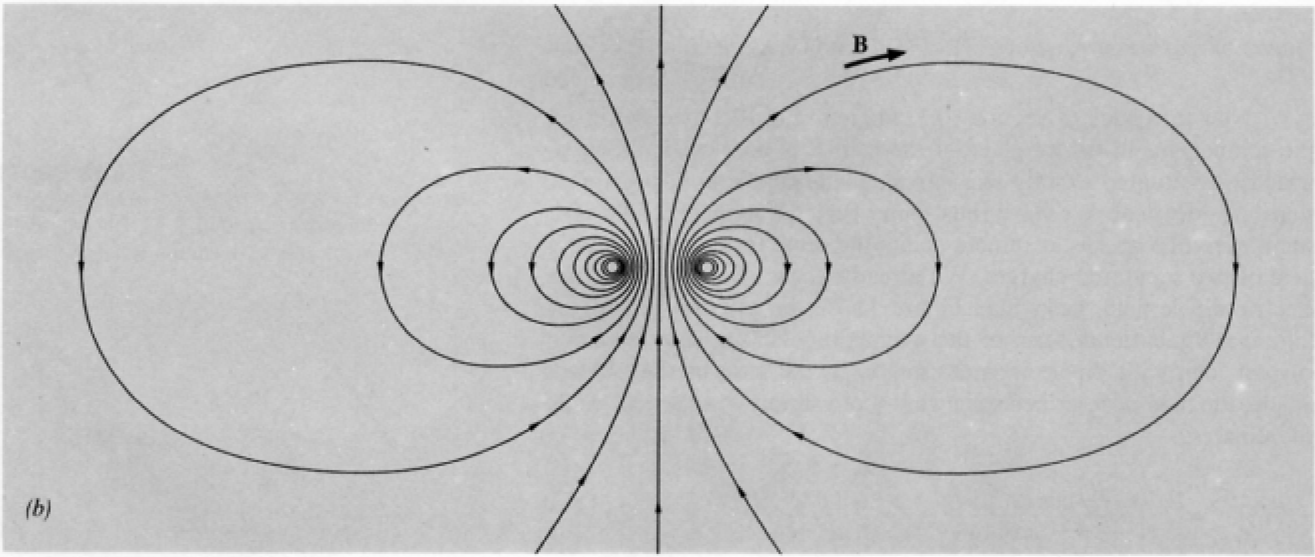


FIGURE 11.8
(*a*) The electric field of a pair of equal and opposite charges. Far away it becomes the field of an electric dipole. (*b*) The magnetic field of a current ring. Far away it becomes the field of a magnetic dipole.

near and *far* we mean, of course, relative to the size of the current loop or the separation of the charges. If we imagine the current ring shrinking in size, the current meanwhile increasing so that the dipole moment $m = Ia/c$ remains constant, we approach the infinitesimal magnetic dipole, the counterpart of the infinitesimal electric dipole described in Chapter 10.