note on rotations

- $$\underset{\sim}{U}(\vec{\theta}) = e^{-\frac{i\vec{\theta}\cdot\vec{\sigma}}{2}} = \begin{pmatrix} \cos\frac{\theta}{2} & -e^{-i\phi}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

rotates state <u>about</u> the $\hat{\theta}$ direction clockwise by amount θ. In the case above, θ and ϕ are chosen to correspond to the spherical polar angles that designate direction of $\langle \underset{\sim}{\vec{S}} \rangle$ (which has magnitude $\hbar/2$).

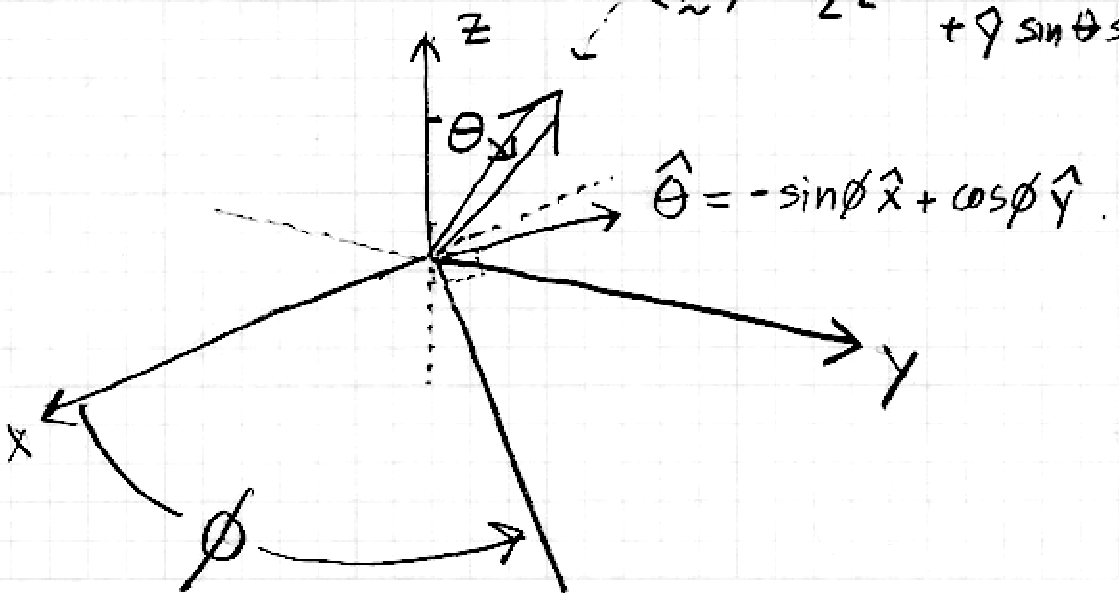
To describe the standard spherical rotation, $\hat{\theta}$ must lie in the x-y plane. That's because θ is the angle between $\langle \underset{\sim}{\vec{S}} \rangle$ and the z-axis. The "right hand rule" or the "rotation <u>about</u> $\hat{\theta}$" indicates that $\hat{\theta}$ must be $\perp$ to the direction described by ϕ.

unit vector in ϕ direction $\Rightarrow (\cos\phi, \sin\phi, 0)$

$\perp$ to that $\Rightarrow (-\sin\phi, \cos\phi, 0)$

Picture (3-d)

$$\langle \underset{\sim}{\vec{S}} \rangle = \frac{\hbar}{2}\left[\hat{z}\cos\theta + \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi\right]$$



- matrix made by state vectors that describe spin-up, down w/r to the direction defined by θ, ϕ differs by phase factors

$$\begin{pmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} & -e^{-i\phi/2}\sin\frac{\theta}{2} \\ e^{i\phi/2}\sin\frac{\theta}{2} & e^{i\phi/2}\cos\frac{\theta}{2} \end{pmatrix} \neq \begin{pmatrix} \cos\frac{\theta}{2} & -e^{-i\phi}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$\begin{pmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} \\ e^{i\phi/2}\sin\frac{\theta}{2} \end{pmatrix} \times e^{i\phi/2} = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}$$

$$\begin{pmatrix} -e^{-i\phi/2}\sin\frac{\theta}{2} \\ e^{i\phi/2}\cos\frac{\theta}{2} \end{pmatrix} \times e^{-i\phi/2} = \begin{pmatrix} -e^{-i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}$$

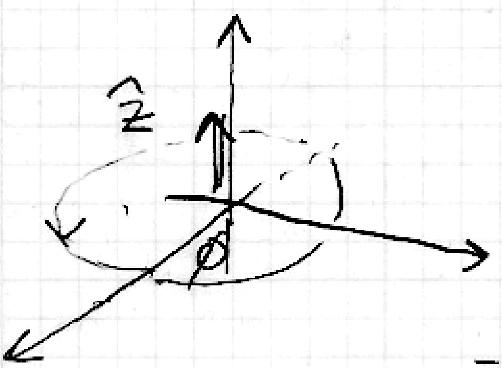
- Two ways to describe rotation about z axis by 2π

① Poor way (from 10/16 class).

- $\theta \simeq \varepsilon$ (small), take $\varepsilon \to 0$
- $\phi = 2\pi$
- use matrix of state vectors.

$$\to \begin{pmatrix} -1 & -1\,\varepsilon/2 \\ -1\,\varepsilon/2 & -1 \end{pmatrix} \to \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

② Better way: simply take $\hat{\theta} = \hat{z}$, rotation angle of ϕ:



$$e^{\frac{-i\phi\,\hat{z}\cdot\vec{\sigma}}{2}} = \cos\frac{\phi}{2} - i\,\underset{\sim}{\sigma}_z \sin\frac{\phi}{2}$$

$$= \begin{pmatrix} \cos\frac{\phi}{2} - i\sin\frac{\phi}{2} & 0 \\ 0 & \cos\frac{\phi}{2} + i\sin\frac{\phi}{2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-\frac{i\phi}{2}} & 0 \\ 0 & e^{\frac{i\phi}{2}} \end{pmatrix} \quad \phi = 2\pi \Rightarrow \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

- rotation about a truly arbitrary direction $\hat{\alpha} = (\alpha_x, \alpha_y, \alpha_z)$ by amount β

$$|\alpha_x|^2 + |\alpha_y|^2 + |\alpha_z|^2 = 1$$

$$\underset{\sim}{U}(\beta\hat{\alpha}) = e^{-\frac{i\beta\hat{\alpha}\cdot\vec{\sigma}}{2}} = \cos\frac{\beta}{2} - i(\hat{\alpha}\cdot\underset{\sim}{\vec{\sigma}})\sin\frac{\beta}{2}$$

$$= \begin{pmatrix} \cos\frac{\beta}{2} & 0 \\ 0 & \cos\frac{\beta}{2} \end{pmatrix} - i\begin{pmatrix} \alpha_z & \alpha_x - i\alpha_y \\ \alpha_x - i\alpha_y & \alpha_z \end{pmatrix} \sin\frac{\beta}{2}$$

$\Rightarrow$ spherical polar $\quad \alpha_x = -\sin\phi$

$\beta = \theta \quad \alpha_y = \cos\phi$

$\alpha_z = 0$

$\Rightarrow$ rotate about z $\quad \beta = \phi$

$\alpha_x = 0$

$\alpha_y = 0$

$\alpha_z = 1$

<u>note</u>:

eigenkets of $\hat{n}\cdot\underset{\sim}{\vec{S}} = \frac{\hbar}{2}\hat{n}\cdot\underset{\sim}{\vec{\sigma}}$

with $\hat{n} = (n_x, n_y, n_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

are eigenkets of $\underset{\sim}{U}(\beta\hat{\alpha})$

with $\beta = \theta$ and $\hat{\alpha} = (-\sin\phi, \cos\theta, 0)$