since those are operators, the ratio is not actually well defined. The spirit is correct. One way to formalize what is happening is to imagine

$$\underset{\sim}{H} = \underset{\sim}{H}^0 + \lambda \underset{\sim}{H}^1$$

$\lambda \Rightarrow$ dimensionless mechanism for power accounting.

and imagine then

$$E_n = E_n^0 + E_n^1 + E_n^2 + E_n^3 + \dots$$

correct if $\lambda = 0$ (order λ^0) — depends on λ^1 — depends on λ^2 — depends on λ^3

(The superscript on the E's is not a power, but a reminder of a power of λ).

$$|n\rangle = |n^0\rangle + |n^1\rangle + |n^2\rangle \text{ etc.}$$

now plug in the expansions to the initial equation

$$\underset{\sim}{H}|n\rangle = E_n|n\rangle$$

$$(\underset{\sim}{H}^0 + \underset{\sim}{H}^1)\{|n^0\rangle + |n^1\rangle + |n^2\rangle + \dots\} = (E_n^0 + E_n^1 + E_n^2 \dots)\{|n^0\rangle + |n^1\rangle + \dots\}$$

0^{th} order only: $\underset{\sim}{H}^0|n^0\rangle = E_n^0|n^0\rangle$

1^{ST} order: $\underset{\sim}{H}^0|n^1\rangle + \underset{\sim}{H}^1|n^0\rangle = E_n^0|n^1\rangle + E_n^1|n^0\rangle$ (*)

(sum of superscripts always 1)

"Bra"-in with $\langle n^0|$

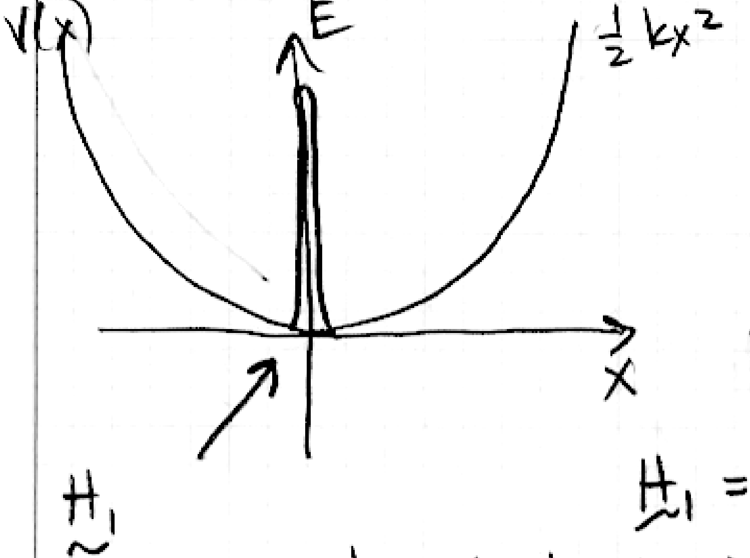
$$\underbrace{\langle n^0|\underset{\sim}{H}^0|n^1\rangle}_{E_n^0\langle n^0|n^1\rangle} + \langle n^0|\underset{\sim}{H}^1|n^0\rangle = E_n^0\langle n^0|n^1\rangle + E_n^1\underbrace{\langle n^0|n^0\rangle}_{1}$$

cancels with term R.H.S.

$$\boxed{E_n^1 = \langle n^0 | \underset{\sim}{H}' | n^0 \rangle}$$

unperturbed unperturbed

Example: 1-d S.H.O. (p.



particle mass m

$\omega = \sqrt{\frac{k}{m}}$

$$\underset{\sim}{H}_0 = \frac{P^2}{2m} + \frac{1}{2}k\underset{\sim}{X}^2$$

$$E_n^0 = (n+\tfrac{1}{2})\hbar\omega$$

$$\underset{\sim}{H}_1 = K\delta(x)$$

$$E_n^1 = \langle n^0 | \underset{\sim}{H}_1 | n^0 \rangle = \int dx\, \psi_n^*(x) K \delta(x) \psi_n(x)$$

$$E_n^1 = K|\psi_n(0)|^2 \qquad \psi_n(0) = 0 \quad n = \text{odd}$$

Got the energy, what about the ket $|n^1\rangle$?

"Bra-in" (*) with $\langle m^0|$, where $m \neq n$

$$\underbrace{\langle m^0 | \underset{\sim}{H}^0 | n^1 \rangle}_{E_m^0 \langle m^0 | n^1 \rangle} + \langle m^0 | \underset{\sim}{H}' | n^0 \rangle = E_n^0 \langle m^0 | n^1 \rangle + E_n^1 \underbrace{\langle m^0 | n^0 \rangle}_{0 \text{ (non-degenerate)}}$$

$$\boxed{\langle m^0 | n^1 \rangle = \frac{\langle m^0 | \underset{\sim}{H}' | n^0 \rangle}{E_n^0 - E_m^0}}$$

(note: natural criteria for estimating "size" of $\underset{\sim}{H}'$: compare with $E_n^0 - E_m^0$.)

determines the projection of $|n^1\rangle$ onto all the $|m^0\rangle$, $m \neq n$.

could say: $|n^1_\perp\rangle \equiv \sum_{m\neq n}\left(\frac{\langle m^0|\underset{\sim}{H}'|n^0\rangle}{E_n^0-E_m^0}\right)|m^0\rangle$

"$\perp$" since $\langle n^0|n^1_\perp\rangle = \sum_{m\neq n}\left(\quad\right)\underbrace{\langle n^0|m^0\rangle}_{0} = 0$

but suppose $|n^1\rangle$ could have a little piece $|n^1_\parallel\rangle$ parallel to $|n^0\rangle$. We can constrain the $|n^1_\parallel\rangle$ by demanding that

$$1 = \langle n|n\rangle = \underbrace{\left(\langle n^0| + \langle n^1_\perp| + \langle n^1_\parallel|\right)\left(|n^0\rangle + |n^1_\perp\rangle + |n^1_\parallel\rangle\right)}_{\text{keep only terms of order 1 or lower.}}$$

normalization still OK

$$1 = \langle n^0|n^0\rangle + \langle n^0|n^1_\perp\rangle + \langle n^1_\perp|n^0\rangle + \langle n^0|n^1_\parallel\rangle + \langle n^1_\parallel|n^0\rangle$$

$$\cancel{1} = \cancel{1} + 0 + 0 + \langle n^0|n^1_\parallel\rangle + \langle n^0|n^1_\parallel\rangle^*$$

conclude: $\langle n^0|n^1_\parallel\rangle$ = purely imaginary $= i\alpha$, α = real #

and so,

$$|n\rangle = \underbrace{|n^0\rangle + i\alpha|n^0\rangle}_{\text{to this order}} + \sum_{m\neq n}\frac{|m^0\rangle\langle m^0|\underset{\sim}{H}'|n^0\rangle}{E_n^0-E_m^0}$$

$= e^{i\alpha}|n^0\rangle$; "re-phase" to $e^{-i\alpha}|n\rangle$

$$\boxed{|n\rangle = |n^0\rangle + \underbrace{\sum_{m\neq n}\frac{|m^0\rangle\langle m^0|\underset{\sim}{H}'|n^0\rangle}{E_n^0-E_m^0}}_{\equiv |n^1\rangle}}$$

influence of $e^{-i\alpha}$ higher order

Go to second order in (✡):

$$\underset{\sim}{H}^0|n^2\rangle + \underset{\sim}{H}^1|n^1\rangle = E_n^0|n^2\rangle + E_n^1|n^1\rangle + E_n^2|n^0\rangle$$

"Bra-in" with $\langle n^0|$

$$\underbrace{\cancel{\langle n^0|\underset{\sim}{H}^0|n^2\rangle}}_{\cancel{E_n^0\langle n^0|n^2\rangle}} + \langle n^0|\underbrace{\underset{\sim}{H}^1|n^1\rangle}_{\sum_{m\neq n}\frac{|m^0\rangle\langle m^0|\underset{\sim}{H}^1|n^0\rangle}{E_n^0-E_m^0}} = \cancel{E_n^0\langle n^0|n^2\rangle} + E_n^1\underbrace{\langle n^0|n^1\rangle}_{0} + E_n^2\underbrace{\langle n^0|n^0\rangle}_{1}$$

0 because $|n^1\rangle$ is all $\perp$ to $|n^0\rangle$

$$E_n^2 = \sum_{m\neq n}\frac{\langle n^0|\underset{\sim}{H}^1|m^0\rangle\langle m^0|\underset{\sim}{H}^1|n^0\rangle}{E_n^0-E_m^0}$$

$$\boxed{E_n^2 = \sum_{m\neq n}\frac{|\langle n^0|\underset{\sim}{H}^1|m^0\rangle|^2}{E_n^0-E_m^0}}$$

Adding a linear term to the S.H.O (1-d)

$$\underset{\sim}{H} = \underset{\sim}{H}^0 + \underset{\sim}{H}^1 = \underbrace{\frac{P^2}{2m} + \frac{1}{2}k\underset{\sim}{x}^2}_{\underset{\sim}{H}^0} \underbrace{- qf\underset{\sim}{x}}_{\underset{\sim}{H}^1}$$

but, $\underset{\sim}{x} = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(\underset{\sim}{a} + \underset{\sim}{a}^+)$

and $\langle n^0|(\underset{\sim}{a} + \underset{\sim}{a}^+)|n^0\rangle = \langle n^0|\left(\sqrt{n}\,|(n-1)^0\rangle + \sqrt{n+1}\,|(n+1)^0\rangle\right)$
$= 0!$

another way: $\langle n^0|\underset{\sim}{x}|n^0\rangle = \int_{-\infty}^{\infty} dx \underbrace{|\varphi_n(x)|^2}_{\text{even}} \underbrace{x}_{\text{odd}} = 0$

$\Rightarrow$ no 1st order energy shift.

However, there is a first-order shift in the eigenket, because:

$$\langle m^0|(\underset{\sim}{a}+\underset{\sim}{a}^\dagger)|n^0\rangle = \sqrt{n}\,\underbrace{\langle m^0|(n-1)^0\rangle}_{\delta_{m,n-1}} + \sqrt{n+1}\,\underbrace{\langle m^0|(n+1)^0\rangle}_{\delta_{m,n+1}}$$

$m \neq n$

$$|n^1\rangle = \sum_{m\neq n} \frac{|m^0\rangle\langle m^0|\underset{\sim}{H}'|n^0\rangle}{E_n^0 - E_m^0}$$

$$= -qf\left(\frac{\hbar}{2m\omega}\right)^{1/2} \sum_{m\neq n} \frac{|m^0\rangle\left(\sqrt{n}\,\delta_{m,n-1}+\sqrt{n+1}\,\delta_{m,n+1}\right)}{\hbar\omega\left(n+\frac{1}{2}-m-\frac{1}{2}\right)}$$

$$\boxed{|n^1\rangle = -qf\left(\frac{1}{2m\hbar\omega^3}\right)^{1/2}\left(+\sqrt{n}\,|(n-1)^0\rangle - \sqrt{n+1}\,|(n+1)^0\rangle\right)}$$

$n-m = n-n+1 = 1$ $n-m = n-m-1 = -1$

so, to <u>second</u> order

$$E_n^2 = \langle n^0|\underset{\sim}{H}_1|n^1\rangle$$

$$= \left[-qf\left(\frac{\hbar}{2m\omega}\right)^{1/2}\right]\left[-qf\left(\frac{1}{2m\hbar\omega^3}\right)^{1/2}\right]\underbrace{\langle n^0|(\underset{\sim}{a}+\underset{\sim}{a}^\dagger)\left\{\sqrt{n}\,|(n-1)^0\rangle - \sqrt{n+1}\,|(n+1)^0\rangle\right\}}_{n[\langle n^0|n^0\rangle]\,-\,(n+1)[\langle n^0|n^0\rangle]}$$

(from $\underset{\sim}{a}^\dagger$)   (from $\underset{\sim}{a}$).

$$\boxed{E_n^2 = -\frac{q^2f^2}{2m\omega^2}}$$

but... we didn't need perturbation theory, actually.

why?

$$\frac{1}{2}kx^2 - qfx = \frac{1}{2}k\left(x^2 - \frac{2qf}{k}x\right)$$

$$= \frac{1}{2}k\left(x - \frac{qf}{k}\right)^2 - \frac{1}{2}k\,\frac{q^2f^2}{k^2}$$

complete the square.

$$= \frac{1}{2}k\left(x - \frac{qf}{k}\right)^2 - \frac{1}{2}\,\frac{q^2f^2}{k}$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$m\omega^2 = k$$

$$\frac{1}{2}kx^2 - qfx = \underbrace{\frac{1}{2}k\left(x - \frac{qf}{k}\right)^2}_{\text{S.H.O, same spring constant as old one}} - \frac{1}{2}\,\frac{q^2f^2}{m\omega^2}$$

shift in energy, EXACT

→ Happens to agree with E_n^2!