Hydrogenic Wave Functions

for $n = 1, 2$

$n = 1, l = 0, m = 0$ $$\psi_{1s} = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_o}\right)^{3/2} e^{-\sigma}$$

$n = 2, l = 0, m = 0$ $$\psi_{2s} = \frac{1}{4\sqrt{2\pi}}\left(\frac{Z}{a_o}\right)^{3/2} (2-\sigma)e^{-\sigma/2}$$

$n = 2, l = 1, m = 0$ $$\psi_{2p_z} = \frac{1}{4\sqrt{2\pi}}\left(\frac{Z}{a_o}\right)^{3/2} \sigma e^{-\sigma/2}\cos\theta$$

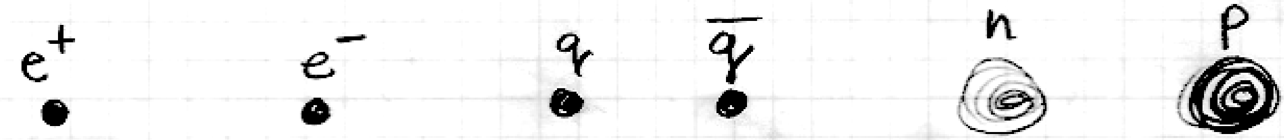
$n = 2, l = 1, m = \pm 1$ $$\psi_{2p_x} = \frac{1}{4\sqrt{2\pi}}\left(\frac{Z}{a_o}\right)^{3/2} \sigma e^{-\sigma/2}\sin\theta \cos\varphi$$

$n = 2, l = 1, m = \pm 1$ $$\psi_{2p_y} = \frac{1}{4\sqrt{2\pi}}\left(\frac{Z}{a_o}\right)^{3/2} \sigma e^{-\sigma/2}\sin\theta \sin\varphi$$

$n = 2, l = 1, m = \pm 1$ $$\psi_{2,1,\pm 1} = \frac{1}{\sqrt{64\pi}}\left(\frac{Z}{a_o}\right)^{3/2} \sigma e^{-\sigma/2}\sin\theta\, e^{\pm i\varphi}$$

$\sigma = Z\,r/a_0$, and a_0 = radius of H, 1s orbital

3 More interesting bound states:



"Positronium" "Quarkonium" "Deuteriom"

All of these systems share the quality that they consist of two particles of similar mass, not one particle moving in the potential created by an infinitely heavy "other".

However, the "two-body" problem, where there is an internal force, reduces down to the one body program. The idea is:

particle #1 $\vec{x}_1, \vec{p}_1$
#2 $\vec{x}_2, \vec{p}_2$
→ Change Variables →

$$\vec{x} = \vec{x}_1 - \vec{x}_2$$

$$\vec{X} = \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_1 + m_2}$$

$$\vec{P} = \vec{p}_1 + \vec{p}_2$$

$$\vec{p} = \mu\left(\frac{\vec{p}_1}{m_1} - \frac{\vec{p}_2}{m_2}\right)$$

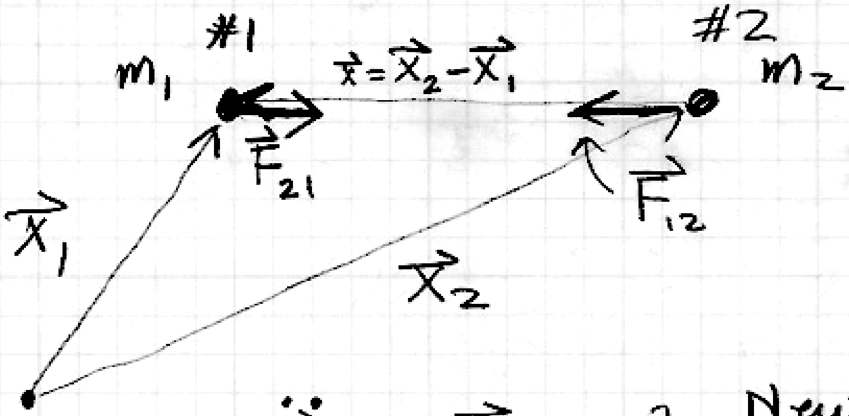
where $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$; $\mu \equiv$ "reduced mass"

$\mu = \frac{m_1 m_2}{m_1 + m_2}$

when the force is internal (exclusively between the two particles), $\vec{X}$ and $\vec{P}$ will (after solving the equation of motion) describe free motion (no force) of a fictitious particle with mass $M = m_1 + m_2$; $\vec{x}$ and $\vec{p}$ describe a fictitious particle of mass μ moving in the internal force.

One description of this change of variables is on pages 85-86 of your text.

The derivation



$$m_1 \ddot{\vec{X}}_1 = \vec{F}_{21}$$
$$m_2 \ddot{\vec{X}}_2 = \vec{F}_{12}$$

Newton's third law: $\vec{F}_{21} = -\vec{F}_{12}$

$$m_1 \ddot{\vec{X}}_1 + m_2 \ddot{\vec{X}}_2 = \vec{F}_{21} + \vec{F}_{12} = 0$$

☆ → $\vec{X} \equiv \dfrac{m_1 \vec{X}_1 + m_2 \vec{X}_2}{m_1 + m_2}$

Then $\ddot{\vec{X}} = 0$ "free" motion is described by $\vec{X}$

Usually: $\vec{F}_{12}$ is a function of $\vec{X}_1 - \vec{X}_2$

☆ → $\vec{x} = \vec{X}_1 - \vec{X}_2$

What next? Invert to get $\vec{X}, \vec{x}$ as a function of $\vec{X}_1, \vec{X}_2$

$$\vec{X}_2 = \vec{X}_1 - \vec{x}$$

$$\vec{X} = \frac{m_1}{m_1+m_2}\vec{X}_1 + \frac{m_2}{m_1+m_2}\vec{X}_2 = \frac{m_1}{m_1+m_2}\vec{X}_1 + \frac{m_2}{m_1+m_2}(\vec{X}_1 - \vec{x})$$

$$= \frac{m_1+m_2}{m_1+m_2}\vec{X}_1 - \frac{m_2}{m_1+m_2}\vec{x} \Rightarrow \vec{X}_1 = \vec{X} + \frac{\mu}{m_1}\vec{x}$$

Then $\vec{x}_2 = \vec{x}_1 - \vec{x} = \vec{X} + \frac{\mu}{m_1}\vec{x} - \vec{x} = \vec{X} + \frac{m_2 - m_1 - m_2}{m_1 + m_2}\vec{x}$

$$\vec{x}_2 = \vec{X} - \frac{m_1}{m_1+m_2}\vec{x} = \vec{X} - \frac{\mu}{m_2}\vec{x}$$

$$m_1\ddot{\vec{x}}_1 = m_1\ddot{\vec{X}} + m_1\left(\frac{\mu}{m_1}\right)\ddot{\vec{x}} = \vec{F}_{21}$$

$$m_1\ddot{\vec{X}} + \mu\ddot{\vec{x}} = \vec{F}_{21} \qquad \text{(\#1)}$$

$$m_2\ddot{\vec{x}}_2 = m_2\ddot{\vec{X}} - m_2\left(\frac{\mu}{m_2}\right)\ddot{\vec{x}} = \vec{F}_{12}$$

$$m_2\ddot{\vec{X}} - \mu\ddot{\vec{x}} = \vec{F}_{12} \qquad \text{(\#2)}$$

(#1) + (#2) $\rightarrow$ $\underline{(m_1+m_2)\ddot{\vec{X}} = 0}$ $\left(\vec{P} = (m_1+m_2)\dot{\vec{X}}\right)$

(#1) − (#2) $\rightarrow$ $2\mu\ddot{\vec{x}} = \vec{F}_{21} - \vec{F}_{12} = 2\vec{F}_{21}$

$$\underline{\mu\ddot{\vec{x}} = \vec{F}_{21}}$$

$$\vec{p} = \mu\dot{\vec{x}} = \mu(\dot{\vec{x}}_1 - \dot{\vec{x}}_2) = \mu\left(\frac{\vec{p}_1}{m_1} - \frac{\vec{p}_2}{m_2}\right)$$

<u>Homework</u>: · another way to connect $\vec{X}, \vec{x}$ to $\vec{P}, \vec{p}$ is with

$$L = T - V$$

then $P_x = \frac{\partial L}{\partial \dot{X}_x}$ $\quad p_x = \frac{\partial L}{\partial \dot{x}_x}$

also assuming: $[\underset{\sim}{X}_{1x}, \underset{\sim}{P}_{1x}] = [\underset{\sim}{X}_{2x}, \underset{\sim}{P}_{2x}] = i\hbar$

show $[\underset{\sim}{X}_x, \underset{\sim}{P}_x] = i\hbar$ $\quad [\underset{\sim}{x}_x, \underset{\sim}{p}_x] = i\hbar$

conclude: transition classical $\longrightarrow$ quantum OK; 2 ficticious particles.

meaning: $\underline{\Psi}(\vec{x}_1, \vec{x}_2) = \psi_{CM}(\vec{X})\psi(\vec{x})$

$$\frac{\vec{P}^2}{2(m_1+m_2)}|\psi_{CM}\rangle = E_{CM}|\psi_{CM}\rangle$$ } work in rest frame

$$\left(\frac{\vec{p}^2}{2\mu} + V(\vec{x}_1 - \vec{x}_2)\right)|\psi\rangle = E|\psi\rangle$$

→ Hydrogen atom:

instead of $a_{0\infty} = \frac{\hbar^2}{m_e e^2} = \frac{\hbar c}{m_e c^2}\frac{\hbar c}{e^2}$

$\hbar c = 1973.3 \text{ eV} \cdot \text{Å} = 1973.3 \cdot 10^{-6} \text{ MeV} \cdot 10^5 \text{ fm}$

(Å ← 10^{-10} m; fm ← 10^{-15} m)

$\hbar c = 197.33 \text{ MeV} \cdot \text{fm}$

$m_e c^2 = 0.5110 \text{ MeV}$

$\frac{\hbar c}{e^2} = \frac{1}{\alpha} = 137.04$

$$a_{0\infty} = \frac{197.33}{0.5110} \times 137.04 \text{ fm}$$

$$a_{0\infty} = 52920 \text{ fm} = 0.5292 \text{ nm}$$

(fm ← 10^{-15} m; nm ← 10^{-9} m)

use: $\mu c^2 = \frac{m_e m_p}{m_e + m_p} c^2 = \frac{(m_e c^2)(m_p c^2)}{(m_e c^2 + m_p c^2)}$

$$= \frac{0.5110 \cdot 938.3}{(0.5110 + 938.3)} \text{ MeV}$$

$$= 0.5110 \cdot \left(\frac{1}{1 + \frac{0.5110}{938.3}}\right) \text{ MeV}$$

$$= 0.5110 \cdot \left(\frac{1}{1 + 0.000545}\right) \text{ MeV}$$