The idea is to factor out the expected behavior in $U_{e\ell}(\hat{\rho})$. It's easiest to do this in 2 steps:

1. $U_{e\ell}(\hat{\rho}) = e^{-\alpha \hat{\rho}} V_{e\ell}(\hat{\rho})$ \{as $\hat{\rho} \to \infty$

$V_{e\ell}(\hat{\rho})$ must not spoil $e^{-\alpha \hat{\rho}}$

$U_{e\ell}^{'}(\hat{\rho}) = -\alpha e^{-\alpha \hat{\rho}} V_{e\ell}(\hat{\rho}) + e^{-\alpha \hat{\rho}} V_{e\ell}^{'}(\hat{\rho})$

$U_{e\ell}^{''}(\hat{\rho}) = \alpha^2 e^{-\alpha \hat{\rho}} V_{e\ell}(\hat{\rho}) - 2\alpha e^{-\alpha \hat{\rho}} V_{e\ell}^{'}(\hat{\rho}) + e^{-\alpha \hat{\rho}} V_{e\ell}^{''}(\hat{\rho})$

plugging back in:

$-\alpha^2 e^{-\alpha \hat{\rho}} V_{e\ell}(\hat{\rho}) + 2\alpha e^{-\alpha \hat{\rho}} V_{e\ell}^{'}(\hat{\rho}) - e^{-\alpha \hat{\rho}} V_{e\ell}^{''}(\hat{\rho})$

$+ \frac{l(l+1)}{\hat{\rho}^2} e^{-\alpha \hat{\rho}} V_{e\ell}(\hat{\rho}) = \frac{2}{\hat{\rho}} e^{-\alpha \hat{\rho}} V_{e\ell}(\hat{\rho}) = \frac{2}{\hat{\rho}} e^{-\alpha \hat{\rho}} V_{e\ell}(\hat{\rho})$

these cancel if we choose $\alpha = -\alpha^2$

since this is a bound state, $\alpha < 0$

and $\alpha = 1 - \frac{2}{l+1}$

canceling $e^{-\alpha \hat{\rho}}$

$V_{e\ell}^{''}(\hat{\rho}) - 2\alpha V_{e\ell}^{'}(\hat{\rho}) + \left[ \frac{2}{\hat{\rho}} - \frac{l(l+1)}{\hat{\rho}^2} \right] V_{e\ell}(\hat{\rho}) = 0$

in general, the next step is to take

$V_{e\ell}(\hat{\rho}) = \hat{\rho}^{l+1} W_{e\ell}(\hat{\rho})$

and $W_{e\ell}(\hat{\rho}) = \sum_{k=0}^{\infty} C_k \hat{\rho}^k$

because of $\hat{\rho}^{l+1}$

but look out: $C_k$ must not spoil $e^{-\alpha \hat{\rho}}$

suppression
\[ V_{\xi \xi}^{l}(\hat{p}) = (l+1) \hat{p}^{l} W_{\xi \xi}^{l}(\hat{p}) + \hat{p}^{l+1} W_{\xi \xi}^{l+1}(\hat{p}) \]
\[ V_{\xi \xi}^{l+1}(\hat{p}) = (l+1) \hat{p}^{l} W_{\xi \xi}^{l+1}(\hat{p}) + 2(l+1) \hat{p}^{l+1} W_{\xi \xi}^{l}(\hat{p}) \]
\[ + \hat{p}^{l+1} W_{\xi \xi}^{l+1}(\hat{p}) \]

Plugging back into the equation:
\[ l(l+1) \hat{p}^{l} W_{\xi \xi}^{l}(\hat{p}) + 2(l+1) \hat{p}^{l} W_{\xi \xi}^{l+1}(\hat{p}) + \hat{p}^{l+1} W_{\xi \xi}^{l+1}(\hat{p}) \]
\[ -2\alpha \{ (l+1) \hat{p}^{l} W_{\xi \xi}^{l}(\hat{p}) + \hat{p}^{l+1} W_{\xi \xi}^{l+1}(\hat{p}) \} \]
\[ + 2 \hat{p}^{l} W_{\xi \xi}^{l}(\hat{p}) - l(l+1) \hat{p}^{l+1} W_{\xi \xi}^{l}(\hat{p}) = 0 \]

Now:
\[ W_{\xi \xi}^{l}(\hat{p}) = \sum_{k=0}^{\infty} C_{k} \hat{p}^{k} \]
\[ W_{\xi \xi}^{l+1}(\hat{p}) = \sum_{k=0}^{\infty} C_{k} \hat{p}^{k-1} \]
\[ W_{\xi \xi}^{l+1}(\hat{p}) = \sum_{k=0}^{\infty} C_{k+1} \hat{p}^{k} \]

Plug in:
\[ 2(l+1) \hat{p}^{l+1} \left[ \sum_{k=0}^{\infty} C_{k+1} \hat{p}^{k} \right] + \hat{p}^{l+1} \left[ \sum_{k=0}^{\infty} C_{k+1} \hat{p}^{k-1} \right] \]
\[ -2\alpha \{ (l+1) \hat{p}^{l} \sum_{k=0}^{\infty} C_{k} \hat{p}^{k} + \hat{p}^{l+1} \sum_{k=0}^{\infty} C_{k} \hat{p}^{k-1} \} \]
\[ + 2 \hat{p}^{l} \sum_{k=0}^{\infty} C_{k} \hat{p}^{k} = 0 \]

Note that as written, every term is of order \( \hat{p}^{l+k} \), so we can collect the coefficients.
\[
\frac{C_{k+1}}{C_k} = 2 \frac{\alpha [k+l+1] - 1}{(k+1)(k+2(l+1))}
\]

so,

\[
\frac{C_{k+1}}{C_k} = 2 \frac{\alpha [k+l+1] - 1}{(k+1)(k+2(l+1))}
\]

1) Can an infinite series work?

In that case,

\[
\lim_{k \to \infty} \frac{C_{k+1}}{C_k} = \frac{2\alpha}{k} \text{ like } e^{+2\alpha\beta} = 1 + 2\alpha\beta + \frac{1}{2}(2\alpha\beta)^2 + \frac{1}{3!}(2\alpha\beta)^3 + \ldots
\]

thus \( U_\infty \sim e^\beta \sum e^{-\alpha \beta} + 2\alpha\beta \) diverges — recall numerical solution.

2) Series will terminate if for some \( k \):

\[
\frac{C_{k+1}}{C_k} = 0 = \alpha [k+l+1] - 1
\]

\[
\alpha = \sqrt{-3} = \frac{1}{k+l+1}
\]

\[
-\frac{E}{E_0} = \frac{1}{(k+l+1)^2}
\]

\[
E = \frac{-1}{(k+l+1)^2} \times E_0 = \frac{-1}{(k+l+1)^2} \times \frac{1}{2} q^2 \alpha^2 \mu c^2
\]
The principal quantum number $n$ is defined

\[ n = k + l + 1 \]  
minimum $n$ is then 0  
no maximum

But there is degeneracy; a variety of $l$ can make the same $n$:

\[ l = 0, 1, 2, \ldots, n-1 \]

The spectrum is often represented the following way:

\[ E = 0 \]
\[ \frac{1}{4} \text{Ry} \]
\[ \frac{1}{4} \text{Ry} \]
\[ \frac{1}{2} \text{Z}^2 \mu^2 \frac{1}{\text{Ry}} \]

\[ n = 3 \quad n = 4 \]
\[ n = 2 \]
\[ n = 1 \]

\[ R_y = \frac{1}{2} \text{Z}^2 \mu^2 \]
\[ l = 0 \quad l = 1 \quad l = 2 \quad l = 3 \]

\[ \text{"s"} \quad \text{"p"} \quad \text{"d"} \quad \text{"f"} \]

note: an infinite # of bound states  
(remember?) since $|V(r)| \to 0$  
slower than $\frac{1}{r^2}$

see p. 35b book.
Wave Functions

\[ R_{\ell \ell}(\hat{\rho}) = \frac{U_{\ell \ell}(\hat{\rho})}{\hat{\rho}} = \hat{\rho} e^{-\alpha \hat{\rho}} W_{\ell \ell}(\hat{\rho}) \]

\[ \alpha = \sqrt{-\varepsilon} = \frac{1}{n} = \sqrt{-\frac{E}{E_0}} \]

\[ E_0 = \frac{\hbar^2}{2a^2} \]

\[ W_{\ell \ell}(\hat{\rho}) = \sum_{k=0}^{n-\ell-1} C_k \hat{\rho}^k \alpha \]

\[ \alpha L^{2l+1}_{n-l-1}(2\hat{\rho}) \]

Remember:

\[ \hat{\rho} = \frac{r}{a} = \frac{z}{a_0} \times r \quad a_0 = \frac{\hbar^2}{\mu e^2} \]

\[ L_0^0 = e^x \left( \frac{d^0}{dx^0} \right) e^{-x} x^p \]

\[ L_p^k = (-1)^k \left( \frac{d^k}{dx^k} \right) L_{p+k}^0 \]

Example:

\[ k = 1 = 2l+1, \quad l = 0 \]

\[ n = 1, \quad p = 0 \]

\[ L_0^1 = (-1)^1 \frac{d}{dx} L_0^0 = -\frac{d}{dx} (1-x) = 1 \]

\[ k = 2 = 2l+1, \quad l = 1 \]

\[ n = 2, \quad p = 2-1-1 = 0 \]

\[ L_0^2 = (-1)^2 \frac{d^2}{dx^2} L_0^0 = \frac{d^2}{dx^2} (2-2x+x^2) = 2 \]

Limiting behavior:

\[ \hat{\rho} \rightarrow \infty \]

\[ R_{\ell \ell}(\hat{\rho}) \sim \hat{\rho}^{\frac{1}{2}} x^{\frac{n-\ell-1}{2}} e^{-\frac{\hat{\rho}}{2n}} - \hat{\rho}^{\frac{1}{2}} x^{\frac{n-\ell-1}{2}} e^{-\frac{\hat{\rho}}{2n}} \]

Normalizations - just look up