The idea is to factor out the expected behavior in $U_{\tilde{\varepsilon}l}(\tilde{\rho})$. It's easiest to do this in 2 steps:

#1 $$U_{\tilde{\varepsilon}l}(\tilde{\rho}) = e^{-\alpha\tilde{\rho}}\, v_{\tilde{\varepsilon}l}(\tilde{\rho})$$ $\left\{\begin{array}{l}\text{as } \tilde{\rho}\to\infty \\ v_{\tilde{\varepsilon}l} \text{ must} \\ \text{not spoil } e^{-\alpha\tilde{\rho}}\text{!}\end{array}\right.$

$$U'_{\tilde{\varepsilon}l}(\tilde{\rho}) = -\alpha e^{-\alpha\tilde{\rho}} v_{\tilde{\varepsilon}l}(\tilde{\rho}) + e^{-\alpha\tilde{\rho}} v'_{\tilde{\varepsilon}l}(\tilde{\rho})$$

$$U''_{\tilde{\varepsilon}l}(\tilde{\rho}) = \alpha^2 e^{-\alpha\tilde{\rho}} v_{\tilde{\varepsilon}l}(\tilde{\rho}) - 2\alpha e^{-\alpha\tilde{\rho}} v'_{\tilde{\varepsilon}l}(\tilde{\rho}) + e^{-\alpha\tilde{\rho}} v''_{\tilde{\varepsilon}l}(\tilde{\rho})$$

plugging back in:

$$-\alpha^2 e^{-\alpha\tilde{\rho}} v_{\tilde{\varepsilon}l}(\tilde{\rho}) + 2\alpha e^{-\alpha\tilde{\rho}} v'_{\tilde{\varepsilon}l}(\tilde{\rho}) - e^{-\alpha\tilde{\rho}} v''_{\tilde{\varepsilon}l}(\tilde{\rho})$$
$$+ \frac{l(l+1)}{\tilde{\rho}^2} e^{-\alpha\tilde{\rho}} v_{\tilde{\varepsilon}l}(\tilde{\rho}) - \frac{2}{\tilde{\rho}} e^{-\alpha\tilde{\rho}} v_{\tilde{\varepsilon}l}(\tilde{\rho}) = \tilde{\varepsilon} e^{-\alpha\tilde{\rho}} v_{\tilde{\varepsilon}l}(\tilde{\rho})$$

these cancel if we choose $\tilde{\varepsilon} = -\alpha^2$

since this is a bound state, $\tilde{\varepsilon} < 0$

and $\boxed{\alpha = \sqrt{-\tilde{\varepsilon}}}$ *

cancelling $e^{-\alpha\tilde{\rho}}$

$$☆ \quad v''_{\tilde{\varepsilon}l}(\tilde{\rho}) - 2\alpha v'_{\tilde{\varepsilon}l}(\tilde{\rho}) + \left[\frac{2}{\tilde{\rho}} - \frac{l(l+1)}{\tilde{\rho}^2}\right] v_{\tilde{\varepsilon}l}(\tilde{\rho}) = 0$$

in general, the next step is to take

$$v_{\tilde{\varepsilon}l}(\tilde{\rho}) = \tilde{\rho}^{\,l+1} w_{\tilde{\varepsilon}l}(\tilde{\rho})$$

and $$w_{\tilde{\varepsilon}l}(\tilde{\rho}) = \sum_{k=0}^{\infty} C_k \tilde{\rho}^k$$ ↙ $k=0$ start because of $\tilde{\rho}^{\,l+1}$

but look out: C_k must not spoil $e^{-\alpha\tilde{\rho}}$ suppression

$$V'_{\tilde{\varepsilon}\ell}(\tilde{\rho}) = (\ell+1)\tilde{\rho}^{\ell} W_{\tilde{\varepsilon}\ell}(\tilde{\rho}) + \tilde{\rho}^{\ell+1} W'_{\tilde{\varepsilon}\ell}(\tilde{\rho})$$

$$V''_{\tilde{\varepsilon}\ell}(\tilde{\rho}) = \ell(\ell+1)\tilde{\rho}^{\ell-1} W_{\tilde{\varepsilon}\ell}(\tilde{\rho}) + 2(\ell+1)\tilde{\rho}^{\ell} W'_{\tilde{\varepsilon}\ell}(\tilde{\rho}) + \tilde{\rho}^{\ell+1} W''_{\tilde{\varepsilon}\ell}(\tilde{\rho})$$

plugging back into the ☆ equation,

$$\cancel{\ell(\ell+1)\tilde{\rho}^{\ell-1} W_{\tilde{\varepsilon}\ell}(\tilde{\rho})} + 2(\ell+1)\tilde{\rho}^{\ell} W'_{\tilde{\varepsilon}\ell}(\tilde{\rho}) + \tilde{\rho}^{\ell+1} W''_{\tilde{\varepsilon}\ell}(\tilde{\rho})$$
$$-2\alpha\left\{(\ell+1)\tilde{\rho}^{\ell} W_{\tilde{\varepsilon}\ell}(\tilde{\rho}) + \tilde{\rho}^{\ell+1} W'_{\tilde{\varepsilon}\ell}(\tilde{\rho})\right\}$$
$$+2\tilde{\rho}^{\ell} W_{\tilde{\varepsilon}\ell}(\tilde{\rho}) - \cancel{\ell(\ell+1)\tilde{\rho}^{\ell-1} W_{\tilde{\varepsilon}\ell}(\tilde{\rho})} = 0 \quad ☆☆$$

now
$$W_{\tilde{\varepsilon}\ell}(\tilde{\rho}) = \sum_{k=0}^{\infty} C_k \tilde{\rho}^{k}$$
$$W'_{\tilde{\varepsilon}\ell}(\tilde{\rho}) = \sum_{k=0}^{\infty} C_k k \tilde{\rho}^{k-1} = \sum_{k=0}^{\infty} C_{k+1}(k+1)\tilde{\rho}^{k}$$
$$W''_{\tilde{\varepsilon}\ell}(\tilde{\rho}) = \sum_{k=0}^{\infty} C_{k+1} k(k+1)\tilde{\rho}^{k-1}$$

plug in ☆☆

$$2(\ell+1)\tilde{\rho}^{\ell}\left[\sum_{k=0}^{\infty} C_{k+1}(k+1)\tilde{\rho}^{k}\right] + \tilde{\rho}^{\ell+1}\left[\sum_{k=0}^{\infty} C_{k+1}k(k+1)\tilde{\rho}^{k-1}\right]$$
$$-2\alpha\left\{(\ell+1)\tilde{\rho}^{\ell}\sum_{k=0}^{\infty} C_k\tilde{\rho}^{k} + \tilde{\rho}^{\ell+1}\sum_{k=0}^{\infty} C_k k\tilde{\rho}^{k-1}\right\}$$
$$+2\tilde{\rho}^{\ell}\sum_{k=0}^{\infty} C_k\tilde{\rho}^{k} = 0$$

note that as written, every term is of order $\tilde{\rho}^{\ell+k}$, so we can collect the coefficients.

$$\left(\underbrace{2(l+1)(k+1)+k(k+1)}_{=(k+1)(2(l+1)+k)}\right)C_{k+1}-2\left(\alpha[l+1+k]-1\right)C_k=0$$

so, $$\boxed{\frac{C_{k+1}}{C_k} = 2\,\frac{\alpha\,[k+l+1]-1}{(k+1)(k+2(l+1))}}$$

1) Can an infinite series work?
In that case,

$$\lim_{k\to\infty}\frac{C_{k+1}}{C_k}=\frac{2}{k}\alpha \quad \text{like } e^{+2\alpha\tilde{\rho}}$$

$$=1+2\alpha\tilde{\rho}+\frac{1}{2}(2\alpha\tilde{\rho})^2+\frac{1}{3!}(2\alpha\tilde{\rho})^3+\ldots$$

thus $$U_{\tilde{\varepsilon}l}\propto\tilde{\rho}^{\,l+1}\underbrace{e^{-\alpha\tilde{\rho}}\,e^{+2\alpha\tilde{\rho}}}_{\text{diverges}}$$

diverges... recall <u>numerical</u> solution.

2) series will terminate if for some k:

$$\frac{C_{k+1}}{C_k}=0=\alpha[k+l+1]-1$$

$$\alpha=\sqrt{-\tilde{\varepsilon}}=\frac{1}{k+l+1}$$

$$-\frac{E}{\tilde{E}_0}=\frac{1}{(k+l+1)^2}$$

$$\boxed{E=\frac{-1}{(k+l+1)^2}\times\tilde{E}_0=\frac{-1}{(k+l+1)^2}\times\frac{1}{2}Z^2\alpha^2\mu c^2}$$

The principal quantum number n is defined

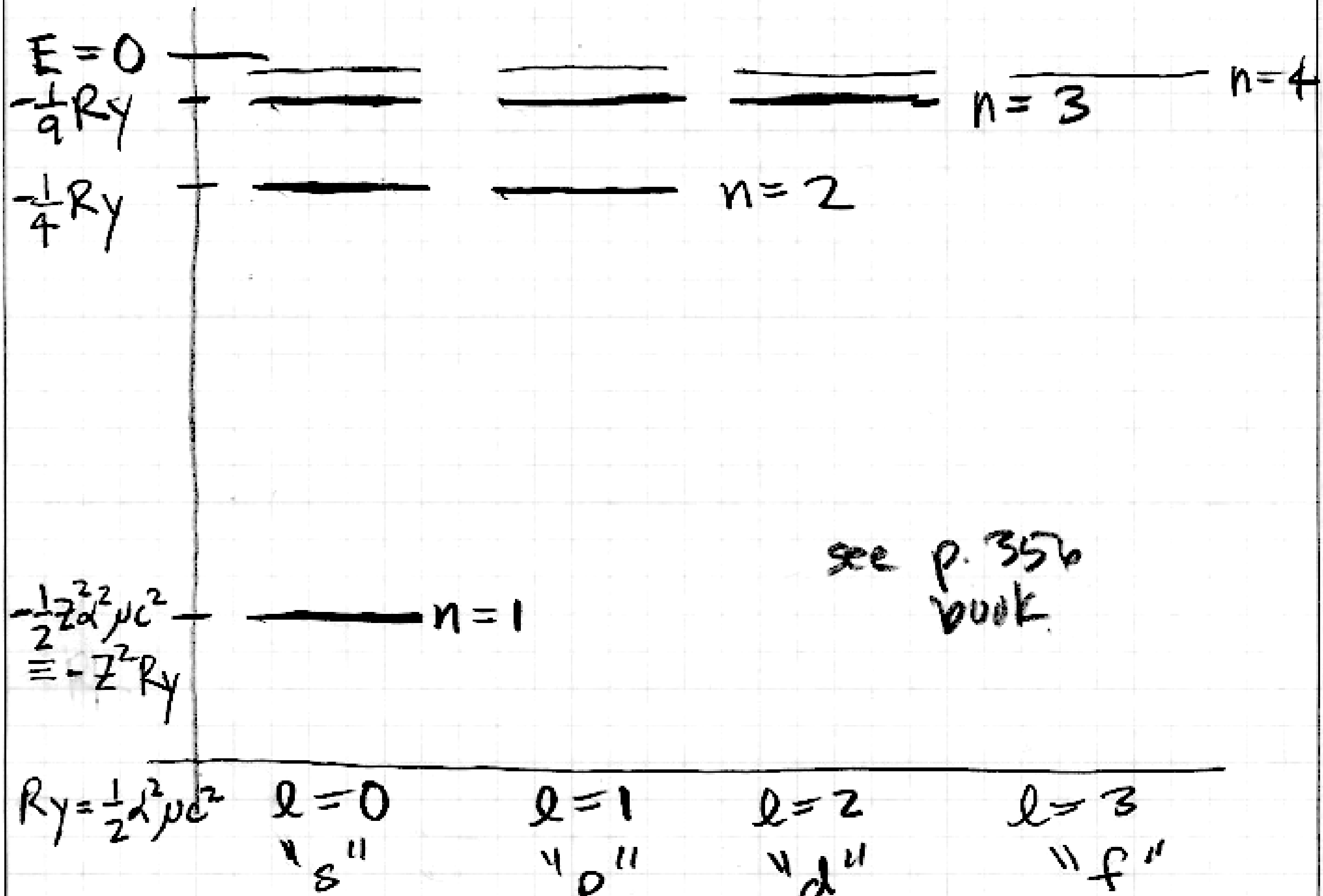
$n = k + l + 1$ minimum n is then $\underline{0}$

no maximum

But there is degeneracy; a variety of l can make the same n:

$$l = 0, 1, 2, \dots, n-1$$

The spectrum is often represented the following way:



note: an ∞ # of bound states (remember?) since $|V(r)| \to 0$ slower than $1/r^2$

Wave Functions lots of back tracking (p. 356 book)

$$R_{\tilde{\varepsilon}l}(\tilde{\rho}) = \frac{U_{\tilde{\varepsilon}l}(\tilde{\rho})}{\tilde{\rho}} = \tilde{\rho}^{l} e^{-\alpha\tilde{\rho}} \, W_{\tilde{\varepsilon}l}(\tilde{\rho})$$

$$\alpha = \sqrt{-\tilde{\varepsilon}} = \frac{1}{n} = \sqrt{-\frac{E}{\tilde{E}_0}} \qquad \tilde{E}_0 = \frac{1}{2} Z^2 \alpha^2 \mu c^2$$

$$W_{\tilde{\varepsilon}l}(\tilde{\rho}) = \sum_{k=0}^{n-l-1} C_k \tilde{\rho}^k \propto \text{"Laguerre Polynomial"}$$

$$\propto L^{2l+1}_{n-l-1}(2\tilde{\rho})$$

remember: $\tilde{\rho} = \frac{r}{\tilde{a}} = \frac{Z}{a_0} \times r \qquad a_0 = \frac{\hbar^2}{\mu e^2}$

$$L^0_p = e^x \left(\frac{d^p}{dx^p}\right) e^{-x} x^p$$

$p = 0$: $= 1$

$= 1$: $e^x[-e^{-x}x + e^{-x}]$, $1 - x$

$= 2$: $e^x[e^{-x}x^2 - 2e^{-x}x + 2e^{-x}]$, $2 - 2x + x^2$

$$\boxed{L^k_p = (-1)^k \left(\frac{d^k}{dx^k}\right) L^0_{p+k}}$$

example: $k = 1 = 2l+1$, $l = 0$
$n = 1$, $p = 0$

$$L^1_0 = (-1)^1 \frac{d}{dx} L^0_1 = -\frac{d}{dx}(1-x) = 1$$

$k = 2 = 2l+1$, $l = 1$
$n = 2$, $p = 2-1-1 = 0$

$$L^2_0 = (-1)^2 \frac{d^2}{dx^2} L^0_2 = \frac{d^2}{dx^2}(2-2x+x^2) = 2$$

... etc

limiting behavior: $\tilde{\rho} \to \infty$

$$R_{nl}(\tilde{\rho}) \sim \tilde{\rho}^{l} \times \tilde{\rho}^{n-l-1} e^{-\frac{\tilde{\rho}}{n}} \sim \tilde{\rho}^{n-1} e^{-\frac{\tilde{\rho}}{n}}$$

Normalizations – just look up