$$\left(\frac{\hbar^2}{2\mu}\left[-\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{r^2}\right]+V(r)\right)U_{E\ell}(r) = E\,U_{E\ell}(r)$$

now, diverge a little from the book (really, just take things in a slightly different order) GENERAL PROPERTIES: 1) ℓ

- $\ell = 0$ .. no orbital angular momentum

Almost just like the 1-d Schrödinger:

$$\left(\frac{-\hbar^2}{2\mu}\frac{d^2}{dr^2}+V(r)\right)U_{E0}(r) = E\,U_{E0}(r)$$

☆ Simplify further: $V(r)=0$!

$$\frac{-\hbar^2}{2\mu}\frac{d^2}{dr^2}U^f_{E0}(r) = E\,U^f_{E0}(r) \qquad f = \text{"free"}$$

with $k^2 \equiv \frac{2\mu E}{\hbar^2}$ it would seem that $U^f_{E0}(r) = Ae^{ikr} + Be^{-ikr}$

$A + B$ ~ arbitrary complex numbers

is a solution

☆ Big point: radial 3-d geometry that includes the origin $(r=0)$ eliminates all but $A=-B$

Why? An odd singularity creeps in; one has to go back to $R^f_{E0}(r)$ to see it:

$$R^f_{E0}(r) = \frac{U^f_{E0}}{r} = \frac{(Ae^{ikr}+Be^{-ikr})}{r}$$

Since $\psi^f_{E0}(\vec{x}) = R^f_{E0}(r)\,Y^0_0(\theta,\phi) = \frac{1}{\sqrt{4\pi}} R^f_{E0}(r)$ is just a multiplicative constant different, we can go back to the full SE:

$$-\frac{\hbar^2}{2m}\nabla^2 R^f_{E0}(r) = E R^f_{E0}(r)$$

$$-\frac{\hbar^2}{2m}\nabla^2\left[\frac{Ae^{ikr}+Be^{-ikr}}{r}\right] = E\left[\frac{Ae^{ikr}+Be^{-ikr}}{r}\right]$$

if one checks both sides of this equation, it almost works; the point $r=0$ has a problem on the left hand side, however:

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\,\delta^{(3)}(\vec{x}) \quad \left\{\text{from Electromagnetism}\right\}$$

the right hand side has nothing like a term such as this. The remedy is to make the coefficient of the $\delta^{(3)}(\vec{x})$ zero near $r=0$, or

$$A+B=0\ , \quad B=-A$$

Conclusion: $U^f_{E0}(r) = A\sin(kr)$ (if $r=0$ present)

- $l \neq 0$ ... $+\frac{l(l+1)}{r^2}$ is a <u>barrier</u> near the origin.. particles pushed <u>out</u>

analyze case where $V(r)$ is negligible compared to $l(l+1)/r^2$ as $r\to 0$

E also becomes unimportant

$$\left[-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2}\right]U_l(r) = 0$$

Try: $U_\ell(r) \propto r^\alpha$ S.E. gives:

$$-\alpha(\alpha-1)+\ell(\ell+1)=0$$

$$\alpha^2-\alpha-\ell(\ell+1)=0$$

$$\alpha=\frac{1\pm\sqrt{1+4\ell(\ell+1)}}{2}=\frac{1\pm(2\ell+1)}{2}$$

$$\alpha=\ell+1,\ -\ell$$

very important

$\boxed{U_\ell(r) \propto r^{\ell+1} \text{ near origin suppression as expected}}$ Type-A

$U_\ell(r) \propto \frac{1}{r^\ell}$ enhancement near origin Type-B ($\ell \geq 1$)

When $r=0$ is in the problem, we keep only Type-A solutions.

Looking back at $\ell=0$, we see that $U_\ell(r) \propto r^1$, the Type-A solution, was in fact selected by the argument we made

For $\ell \geq 1$,

The enhancement at the origin is actually (physically) sufficient to eliminate Type-B. An additional flaw, that Type-B solutions result in a non-Hermitian Hamiltonian, is detailed on p.341 of text.

<u>Note</u>: if $r=0$ is not to be considered, Type-II solutions are fine!

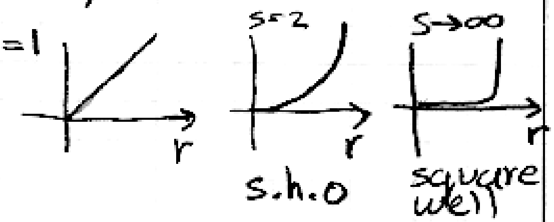
GENERAL PROPERTIES : 2) Length Scale, Power Law Potentials

Power Law Potential:

$$V(r) = V_0 r^s$$

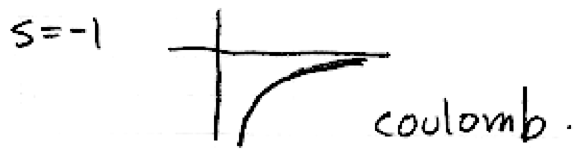
s = power

- attractive: a) $V_0 > 0$ $\quad s > 0$



- dimensions of V_0 are $\frac{\text{Energy}}{\text{Length}^s}$

b) $V_0 < 0$ $\quad s < 0$



- repulsive: reverse V_0 sign

$$\left\{\frac{\hbar^2}{2\mu}\left[-\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2}\right] + V_0 r^s\right\} U_{E\ell}(r) = E\, U_{E\ell}(r)$$

define a length scale $\underline{\underline{a}}$, so a dimensionless radius $\rho \equiv \frac{r}{a}$; $r = a\rho$

plugging back in

$$\left\{\frac{\hbar^2}{2\mu a^2}\left[-\frac{d^2}{d\rho^2} + \frac{\ell(\ell+1)}{\rho^2}\right] + V_0 a^s \rho^s\right\} U_{E\ell}(\rho) = E\, U_{E\ell}(\rho)$$

Idea: choose a so that these coefficients are equal.

Physics: a is where radial contribution to kinetic energy = potential energy.

$$\frac{\hbar^2}{2\mu a^2} = |V_0| a^s \Rightarrow \boxed{a = \left(\frac{\hbar^2}{2\mu |V_0|}\right)^{\frac{1}{s+2}}}$$

- $s = 2$
  $|V_0| = \frac{1}{2}\mu\omega^2$

$$a = \left(\frac{\hbar^2}{2\mu \cdot \frac{1}{2}\mu\omega^2}\right)^{\frac{1}{4}} = \sqrt{\frac{\hbar}{m\omega}}$$

see p. 191, 7.3.6

- $s = -1$
  $|V_0| = Ze^2$
  (charge Z)

$$a = \left(\frac{\hbar^2}{2\mu Ze^2}\right)^{\frac{1}{-1+2}}$$

$$= \frac{1}{2Z}\frac{\hbar^2}{\mu e^2} = \frac{1}{2Z} a_0$$

$a_0 \rightarrow$ Bohr radius
see p. 357, 13.1.24

- $s \rightarrow \infty$ $\quad a \rightarrow \left(\frac{\hbar^2}{2\mu |V_0|}\right)^{\frac{1}{\infty+2}} \rightarrow 0$

but $|V_0|$ can be simultaneously adjusted to make a anything.

What is the value of the common coefficents?

$$\frac{\hbar^2}{2\mu a^2} = |V_0| a^s = |V_0| \left(\frac{\hbar^2}{2\mu |V_0|}\right)^{\frac{s}{s+2}} \equiv E_0$$

in these terms, the radial equation now becomes:

$$\boxed{\left\{ -\frac{d^2}{d\rho^2} + \frac{\ell(\ell+1)}{\rho^2} + \operatorname{sgn}(V_0)\, \rho^s \right\} U_{\varepsilon\ell}(\rho) = \varepsilon\, U_{\varepsilon\ell}(\rho)}$$

"dimensionless radial equation"

$+1$ when $V_0 > 0$

-1 when $V_0 < 0$