Big ideas:

- 1-2 (but can be used well in spherically symmetric situations, e.g., $\alpha$-decay).
- When $V(x)$ changes "slowly".
- "Wavey"... describes tunnelling of particles "under" high potentials.

\[ \psi(x) = A e^{\pm i k x} \]

\[ E = \frac{p^2}{2m} = E - V \]

\[ \hbar k = p = \sqrt{2m(E-V)} \]

\[ k = \frac{1}{\hbar} \sqrt{2m(E-V)} \]

$kx \rightarrow "phase advance" \text{ in distance } x$...
Suppose the following situation:

\[ E \quad \text{(constant)} \]

Kinetic: \[ -E - V(x) = \frac{p^2}{2m} \]

\[ V(x) \quad \text{not constant} \]

\[ \text{maybe} \quad (\hbar k(x))^2 = p^2 = 2m(E-V(x)) \]

\[ k(x) = \frac{1}{\hbar} \sqrt{2m(E-V(x))} \]

Phase advance: instead of \( kx \)

\[ = \int dx \, k(x) \]

\[ \psi(x) = Ae^{\pm i\int dx \, k(x)} = Ae^{\pm \frac{i}{\hbar} \int \text{S}(x) dx} \]

Lowest order dependence of \( A \) on \( x \)
Physically, expect

\[ |\psi(x)|^2 \propto \frac{1}{\sqrt{V(x)}} \quad \text{velocity} \]

but \[ V(x) = \frac{p(x)}{m} \implies \psi^2 \propto \frac{1}{p} \]

so,

\[ \psi(x) = \frac{C}{\sqrt{p(x)}} e^{-\frac{i}{\hbar} \int p(x) dx} \]

\[ p(x) = \sqrt{2m(E-V(x))} \]

\[ \Rightarrow \text{formal derivation pp. 316/317} \]

\[ \Rightarrow (\text{Qualitatively decreasing wavelength, increasing amplitude}) \]

\[ \Rightarrow 2\pi \rightarrow 1 \quad \text{phase advance} \]

\[ \text{Increasing wavelength, decreasing amplitude} \]

\[ \text{Constant } V \]
"Trapped" in a well.

\[ V(x) \]

not a constant.

\[ \psi(x) = \frac{1}{\sqrt{p(x)}} \left[ C_+ e^{i \frac{1}{\hbar} \int p(x) dx} + C_- e^{-i \frac{1}{\hbar} \int p(x) dx} \right] \]

\[ \psi(0) = \psi(a) = 0! \]

with a little rearrangement,

\[ \psi(x) \equiv \frac{1}{\sqrt{p(x)}} \left[ C_1 \sin \left( \frac{1}{\hbar} \int p(x) dx \right) + C_2 \cos \left( \frac{1}{\hbar} \int p(x) dx \right) \right] \]

\[ i (C_+ - C_-) \left[ \frac{e^i - e^{-i}}{2i} \right] \]

\[ (C_+ + C_-) \left[ \frac{e^i + e^{-i}}{2} \right] \]

\[ \frac{1}{\hbar} \int_0^a p(x) dx = n \pi \]

\[ n = 1 \text{ (half-wave)} \]

2 full wave

Utility... quantum wells.
Don’t dwell on this… even more interesting is Tunneling...

\[ E \]

\[ V(x) \]

\[ A e^{\frac{j \phi}{\hbar}} x \rightarrow e^{\frac{j \phi}{\hbar}} x \]

\[ V(x) = 0 \]

\[ x = 0 \]

Incident: \( A e^{\frac{j \phi}{\hbar}} x \)

Reflection: \( B e^{\frac{j \phi}{\hbar}} x \)

What next? Reflection \( B e^{\frac{j \phi}{\hbar}} x \)

and “negative kinetic energy”\( T = E - V(x) < 0 \)

\[ = \frac{p^2}{2m} < 0 \]

OK if \( p \) imaginary

then “under barrier” solutions

\[ \sqrt{\rho(p(x))} Ce^{\frac{1}{\hbar} \int |p(x)| dx} + De^{-\frac{1}{\hbar} \int |p(x)| dx} \]
The first term is a rising exponential, second a dying one.
For a wave incident from left, only a dying exponential should be present.

Then:
\[ P(x) \]
\[ P(x) \]
\[ x = 0 \]
\[ x = \text{much smaller} \]

\[
\frac{|P(x)|}{|A|} \sim e^{-\frac{1}{h} \int |P(x)| dx}
\]

\[
\frac{1}{\sqrt{P(x)}} \text{ neglected}
\]

\[ T = \frac{|P|^2}{|A|^2} = e^{-2\theta}
\]

\[ \gamma = \frac{1}{h} \int_0^a |P(x)| dx
\]
Unstable Nuclei

Some nuclei are actually unstable to breaking into smaller fragments.

In earth:

\[ ^{238}_{92}U \rightarrow ^{234}_{90}Th + \gamma \]

\[ ^{232}_{90}Th \rightarrow ^{228}_{88}Ra + ^4He \]

\[ ^{235}_{92}U \rightarrow ^{231}_{90}Th + ^4He \]

Absence of shorter-lived isotopes - EARTH's AGE

Many shorter lived cases exist:

1) In decay chain of above:
   \[ \rightarrow ^{222}_{86}Ra \quad (T_{1/2} \approx 1600 \text{ y}) \]

2) Artificial Produced
   \[ \rightarrow ^{239}_{94}Pu \quad (T_{1/2} \approx 24,000 \text{ y}) \]
Idea: $Z = 90$  

$^{234} Th \rightarrow r \rightarrow ^{4} He$ (both at rest)  

LESS ENERGETIC THAN  

$^{238} U$  

What keeps $^{234} Th + ^{4} He$ together?  

STRONG FORCE ...  

$U(r)$  

$\not \sim 0$.  

stable  

$\sim 1/r$  

tunnels at