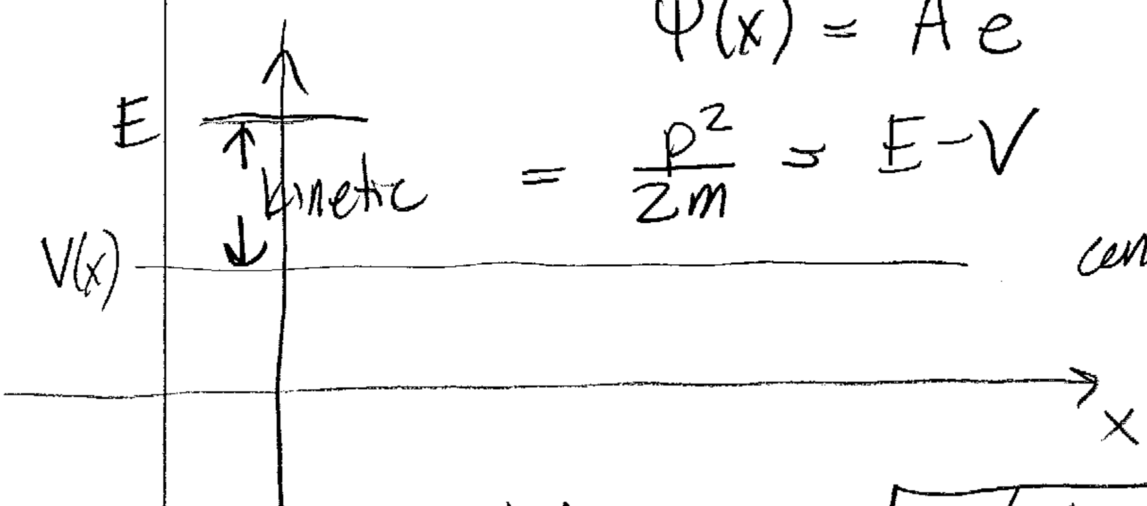
WKB Approximation — Chapter 8

Big ideas:

- 1-d (but can be used well in spherically symmetric situations... α-decay).
- when $V(x)$ changes "slowly"
- "Wavey"... describes tunnelling of particles "under" high potentials

$$\psi(x) = A e^{\pm ikx}$$



$$\text{kinetic} = \frac{p^2}{2m} = E - V$$

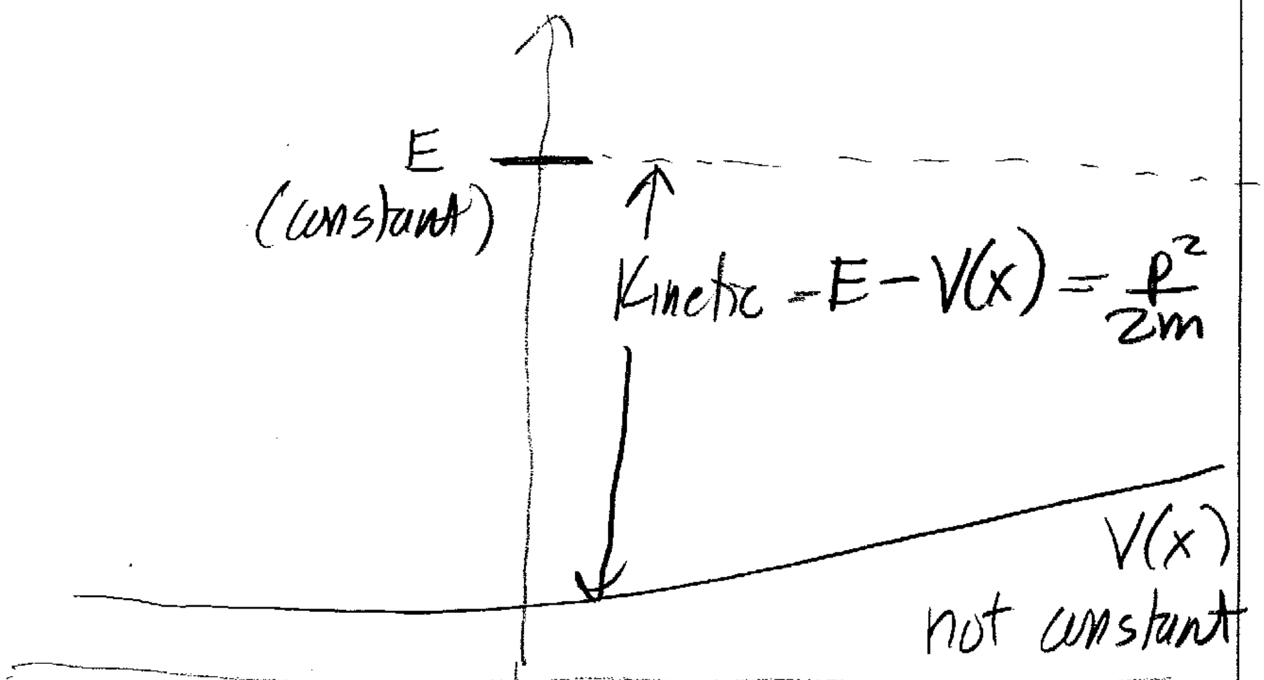
constant

$$\hbar k = p = \sqrt{2m(E-V)}$$

$$k = \frac{1}{\hbar}\sqrt{2m(E-V)}$$

$kx \rightarrow$ "phase advance" in distance x...

Suppose Following Situation:



maybe $\left(\hbar k(x)\right)^2 = p^2 = 2m\left(E - V(x)\right)$

$$k(x) = \frac{1}{\hbar}\sqrt{2m\left(E - V(x)\right)}$$

phase advance .. instead of kx

$$= \int dx\, k(x)$$

$$\psi(x) \simeq A e^{\pm i \int dx\, k(x)} = A e^{\pm \frac{i}{\hbar}\int p(x)\, dx}$$

Lowest order dependence of A on x

Physically, expect

$$|\psi(x)|^2 \propto \frac{1}{v(x)} \leftarrow \text{velocity}$$

but $v(x) = \frac{p(x)}{m} \Rightarrow \psi^2 \propto \frac{1}{p}$

$\psi \propto \frac{1}{\sqrt{p}} e^{\pm \frac{i}{\hbar}\int p(x)dx}$

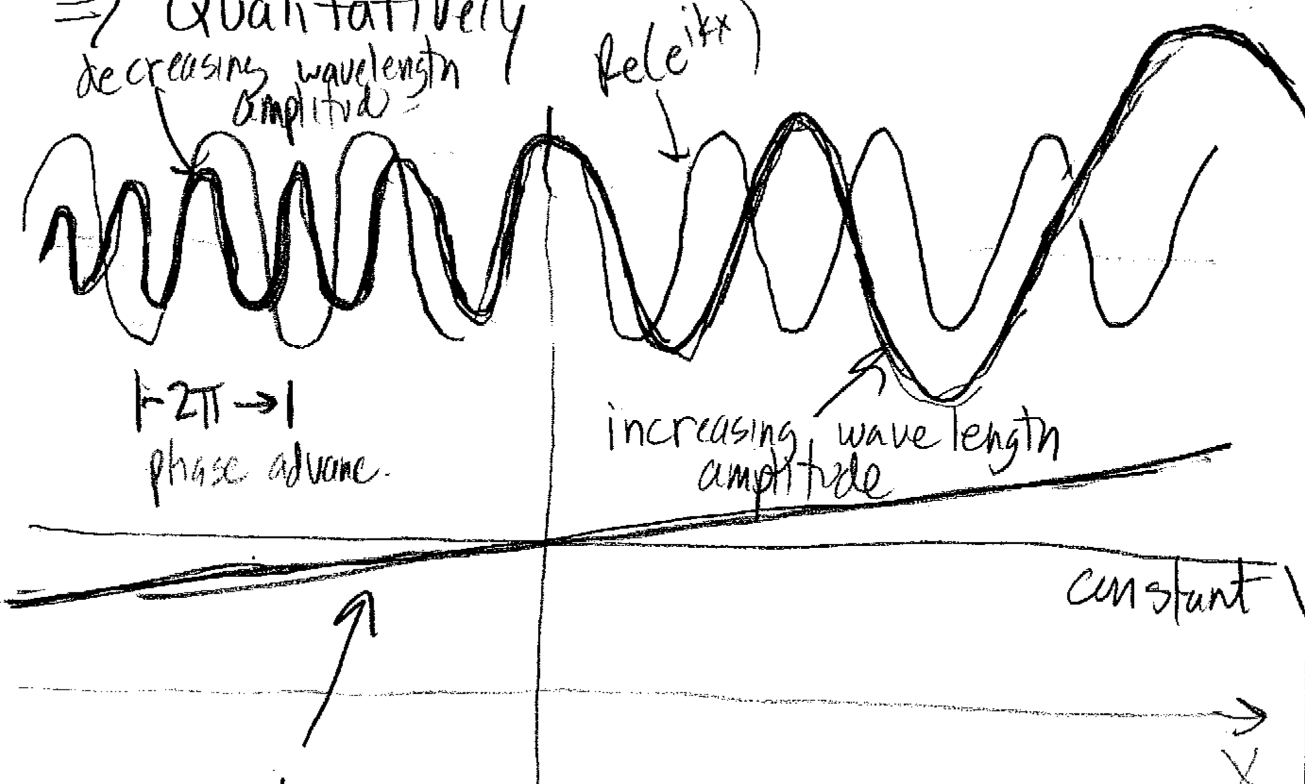
so,

$$\boxed{\psi(x) = \frac{C}{\sqrt{p(x)}} e^{\pm \frac{i}{\hbar}\int p(x)dx} \qquad p(x) = \sqrt{2m(E - V(x))}}$$

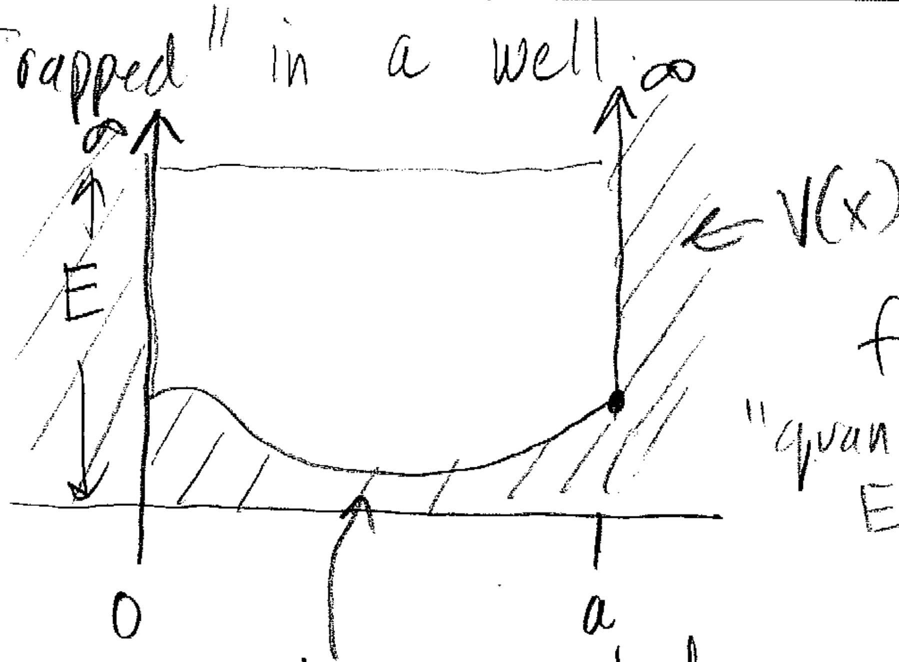
also, simple norm.

⇒ formal derivation pp. 316/317

⇒ Qualitatively



"Trapped" in a well



find "quantized" E_n

$$\psi(x) \simeq \frac{1}{\sqrt{p(x)}}\left[C_+ e^{\frac{i}{\hbar}\int p(x)dx} + C_- e^{-\frac{i}{\hbar}\int p(x)dx}\right]$$

$$\psi(0) = \psi(a) = 0 \, !$$

with a little rearrangement,

$$\psi(x) \cong \frac{1}{\sqrt{p(x)}}\left[\underset{\uparrow \atop i(C_+ - C_-)\left[\frac{e^+ - e^-}{2i}\right]}{C_1} \sin\left(\frac{1}{\hbar}\int p(x)dx\right) + \underset{\uparrow \atop (C_+ + C_-)\left[\frac{e^+ + e^-}{2}\right]}{C_2} \cos\left(\frac{1}{\hbar}\int p(x)dx\right)\right]$$

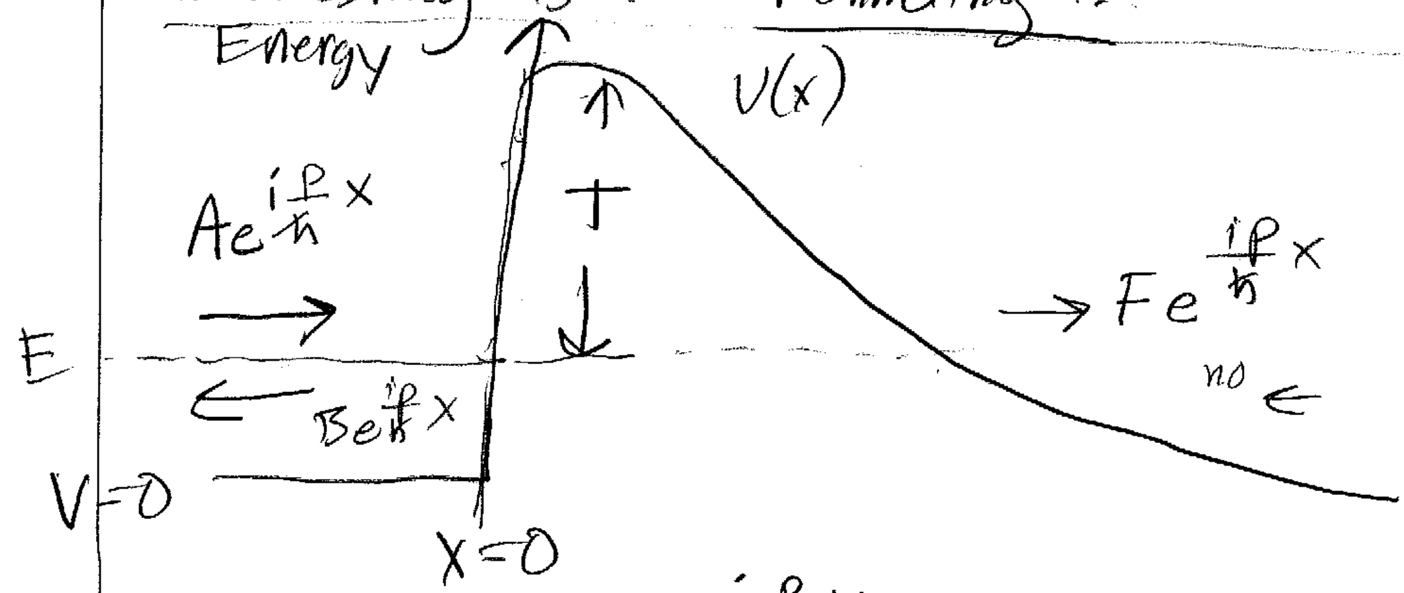
$$\boxed{\frac{1}{\hbar}\int_0^a p(x)\,dx = n\pi}$$

$C_2 = 0$!
$\psi(0) = 0$

$n = 1$ (half-wave)
2 full wave

Utility.. quantum wells.

Don't dwell on this... even more interesting is... Tunneling...



Incident: $Ae^{i\frac{p}{\hbar}x}$

What next? Reflection $Be^{-i\frac{p}{\hbar}x}$

and "negative kinetic energy"

$$T = E - V(x) < 0$$

$$= \frac{p^2}{2m} < 0$$

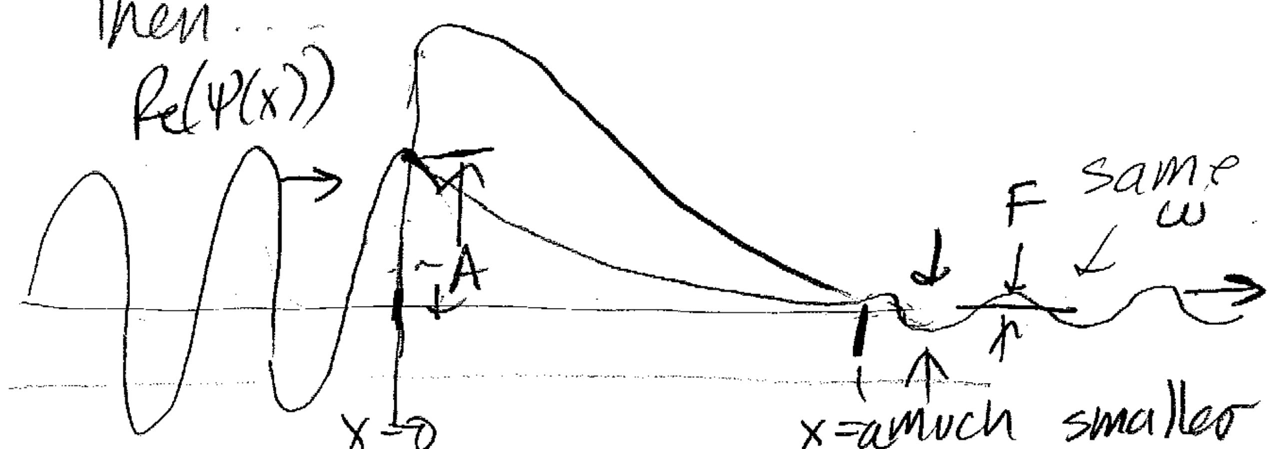
OK if p imaginary

then, "under barrier", solutions look like

$$\frac{1}{\sqrt{|p(x)|}}\left(Ce^{+\frac{1}{\hbar}\int_0^x |p(x')|dx'} + De^{-\frac{1}{\hbar}\int_0^x |p(x')|dx'}\right)$$

The first term is a rising exponential, second a dying one ..
For a wave incident from left, only dying exponential should be present.

Then ...



$$\frac{|F|}{|A|} \sim e^{-\frac{1}{\hbar}\int_0^a |p(x')|\,dx'}$$

$\frac{1}{\sqrt{p(x)}}$ neglected.

$$\text{or} \quad T = \frac{|F|^2}{|A|^2} = e^{-2\gamma}$$

$$\gamma = \frac{1}{\hbar}\int_0^a |p(x)|\cdot dx$$

Unstable Nuclei

Some nuclei are actually unstable to breaking into smaller fragments.

In earth:

$\tau_{1/2} = 4.5 \cdot 10^9 y$ $\left\{ {}^{238}U \rightarrow {}^{234}Th + \underbrace{{}^{4}He}_{\alpha\text{-decay}} \right.$

$Z = 92 \qquad Z = 90$

$\tau_{1/2} = 14 \cdot 10^9 y$ $\left\{ {}^{232}Th \rightarrow {}^{228}Ra + {}^{4}He \right.$

$Z = 90 \qquad Z = 88$

$.7 \cdot 10^9 y$ $\left\{ {}^{235}U \rightarrow {}^{231}Th + {}^{4}He \right.$

$Z = 92 \qquad Z = 90$

Absence of shorter-lived isotopes — EARTH's AGE

Many shorter lived cases exist:

1. In decay chain of above $\Rightarrow {}^{222}Ra \quad (\tau_{1/2} \simeq 1600\, y)$
2. Artificial Produced $\Rightarrow {}^{239}Pu \quad (\tau_{1/2} \simeq 24{,}000\, y)$

Idea...

$Z=90$ $\qquad$ $Z=2$

$^{234}Th \leftarrow r\to\infty \rightarrow He$ (both at rest)

LESS ENERGETIC THAN

^{238}U !!!

What keeps ^{234}Th + He together?

STRONG FORCE...