Hydrogen + Fine Structure

$\underbrace{\text{Fine}}_{\text{aka Relativistic Perturbations}}$

aka Relativistic Perturbations

Review (unperturbed)

$$E_n = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2\right]\frac{1}{n^2}$$

l: from 0 $n-1$

NO DEPENDENCE ON l in E_n

p. 149, eq 4.70 $\quad n = 1, 2, \ldots \infty$

Another way to write this (important)

$$E_n = -\frac{1}{2}\underbrace{\left(\frac{e^2}{4\pi\varepsilon_0\hbar c}\right)}_{\text{fine structure constant } \alpha \cong 1/137}^2 mc^2 \times \frac{1}{n^2}$$

$$\boxed{E_n = -\frac{1}{2}\alpha^2 mc^2 \times \frac{1}{n^2}}$$

technically $m = \dfrac{m_e m_p}{m_e + m_p}$

Think of $\dfrac{v_e}{c} = \dfrac{\alpha}{nc}$, $E_n \simeq -\dfrac{1}{2} m v_e^2$

Actually, virial theorem, p. 190 Eq 4.191

$$\langle T \rangle = -E_n = +\frac{1}{2} m v_e^2$$

$$\langle V \rangle = 2E_n \quad \text{now note} \ldots$$

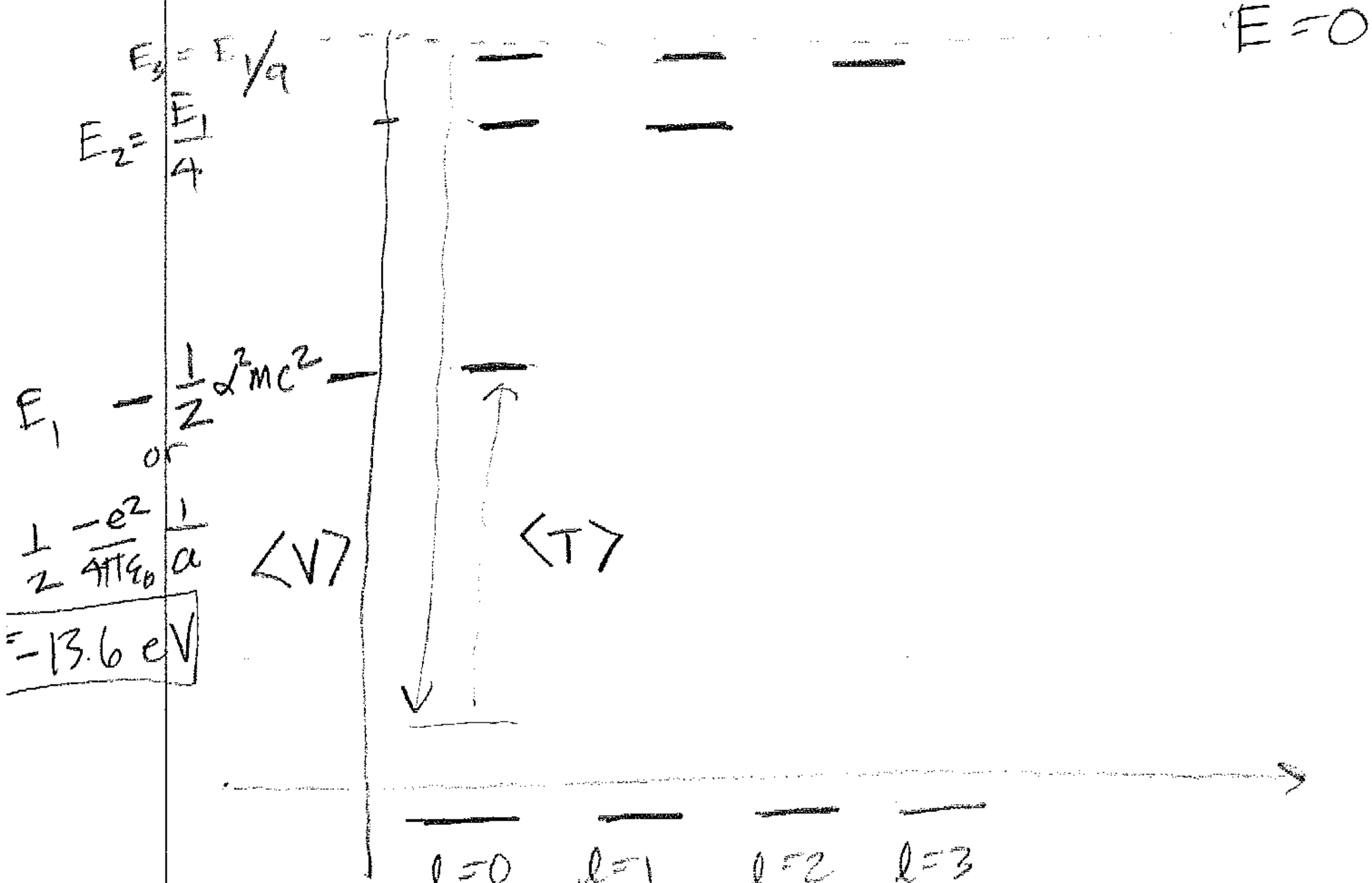
$$2E_n = -\frac{e^2}{4\pi\varepsilon_0} \cdot \frac{1}{\underbrace{4\pi\varepsilon_0 \frac{\hbar^2}{me^2}}_{\text{Bohr radius } a}} \cdot \frac{1}{n^2}$$

note: $a = \dfrac{1}{\alpha} \cdot \lambda$

$= \dfrac{4\pi\varepsilon_0\hbar c}{e^2} \quad \dfrac{\hbar}{mc}$

$$2E_n = -\frac{e^2}{4\pi\varepsilon_0}\frac{1}{a}\cdot\frac{1}{n^2}$$

For $\frac{-e^2}{4\pi\varepsilon_0 r}$ potential...



Relativistic Corrections

Griffiths... always m

$$O\left(\frac{E_n^2}{mc^2}\right) \Rightarrow O\left(\frac{\cancel{m^2} v_e^4}{\cancel{m}c^2 n^4}\right) \Rightarrow O\left[\left(\frac{v_e}{nc}\right)^4 \cdot m_e c^2\right]$$

$$O\cdot\left[\left(\frac{1}{137}\right)^2 \cdot 13.6\right]$$

$$\updownarrow$$

$$\underbrace{\sim 5\cdot 10^{-5}}_{\text{"resolution" needed}} \times 13.6 \sim 7\cdot 10^{-4}\ \text{eV}$$

or, $\lambda = \frac{2\pi\hbar c}{E} \sim \frac{2\pi \cdot 197.3 \text{ eV}\cdot\text{nm}}{7\cdot 10^{-4}} \sim 1.8 \text{ mm}$

More specifically, the precise prefactor needed is:

$$\frac{2(E_n^2)}{mc^2} = \frac{2\cdot\frac{1}{4}\alpha^4 m^2c^4}{mc^2\, n^4} = \frac{\alpha^2}{n^2}\left[\frac{1}{2}\frac{\alpha^2 mc^2}{n^2}\right]$$

$$= \frac{\alpha^2}{n^2}|E_n|$$

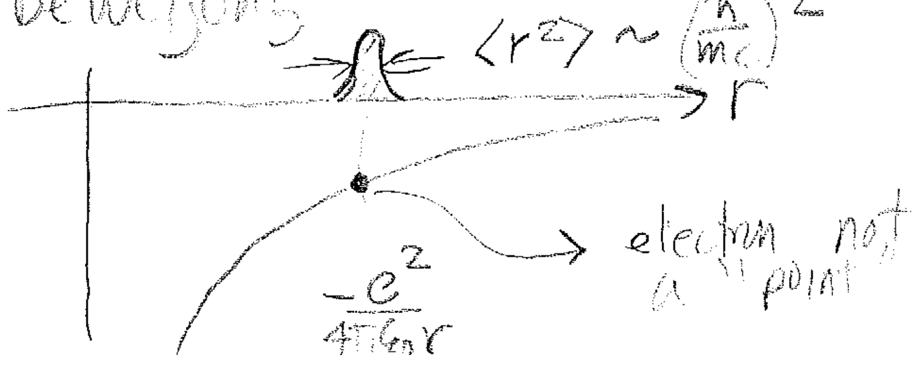
Ⓐ $$H_r' = -\frac{P^4}{8m^3c^2} \Rightarrow E_r' = \alpha^2|E_n|\underbrace{\left[-\frac{1}{n^2}\left[\frac{n}{\ell+\frac{1}{2}} - \frac{3}{4}\right]\right]}_{\varepsilon_r'(n,\ell)}$$

$\varepsilon_r'(n,\ell)$ (↑ slower)

	$\ell=0$	$\ell=1$	$\ell=2$	$\ell=3$	$\ell=4$
$n=5$	-0.370	-0.103	-0,050	-0.0271	-0,0144
$n=4$	-0,453	-0.120	-0.053	-0.0246	
$n=3$	-0,583	-0,139	-0,050		
$n=2$	-0,813	-0.146			
$n=1$	-1,25				

slower →

Ⓑ Neglected in text... the "Darwin" Term... physics is the electron is "spread out" over a distance of $\lambda \sim \frac{\hbar}{mc}$ "Zitterbewegung"



Darwin Term Discussion

Note (p. 270, Eq. 6.57) the $O(p^4)$ shift is..

$$E_r^1 = -\frac{1}{2}\frac{(E_n^2)}{mc^2}\left[\frac{4n}{l+\frac{1}{2}} - 3\right]$$

when $l=0$ (s-wave)

$$\underline{\underline{E_r^1 = -\frac{1}{2}\frac{(E_n^2)}{mc^2}\left[8n-3\right]}} \quad (\alpha)$$

now, compare with (p. 274, Eq. 6.66), the <u>complete</u> $O\left(\frac{E_n^2}{mc^2}\right)$ shift

$$E_{fs}^1 = -\frac{1}{2}\frac{(E_n^2)}{mc^2}\left[\frac{4n}{j+1/2} - 3\right]$$

superficially, resembles Eq. 6.57, with l replaced by j. This is a big deal, though.. when $l=0$, $j=s=\frac{1}{2}$ (spin of electron).

Now...

$$\underline{\underline{E_{fs}^1 = -\frac{1}{2}\frac{E_n^2}{mc^2}\left[4n-3\right]}} \quad (\beta)$$

<u>Main point</u>: This difference between (α) and (β) cannot be due to the spin-orbit interaction, because $l=0$ so $\langle\vec{L}\cdot\vec{S}\rangle=0$. The difference between (α) + (β) is caused by yet another physical effect, "The Darwin Term".

<u>Physics underlying the Darwin Term</u>...

the electron is "smeared out" over a distance scale of approximately its Compton Wavelength (reduced).

Compton Wavelength (reduced) $\equiv \frac{\hbar}{mc} \equiv \lambda\!\!\!^{-}$

note: $\lambda\!\!\!^{-} = \frac{\hbar}{mc} = \underbrace{\frac{e^2}{4\pi\varepsilon_0 \hbar c}}_{\substack{\text{fine structure constant } \alpha \\ \text{p. 267 Eq. 6.43}}} \cdot \underbrace{\frac{\hbar^2}{me^2} \cdot 4\pi\varepsilon_0}_{\substack{\text{Bohr Radius } a \\ \text{p. 150 Eq. 4.72}}}$

$$\lambda\!\!\!^{-} = \alpha a \qquad \alpha \simeq \frac{1}{137}$$

Because the electron is spread out, it doesn't quite feel the $1/r$ potential ... instead ...

$$\underbrace{\langle V_{eff}(\vec{r}) \rangle}_{\text{averaged over electron spread}} \simeq \underbrace{-\frac{e^2}{4\pi\varepsilon_0} \frac{1}{r}}_{\text{point particle}} + \frac{1}{2}\left(\left\langle x^2 \frac{\partial^2 V}{\partial x^2} \right\rangle + \left\langle y^2 \frac{\partial^2 V}{\partial y^2} \right\rangle + \left\langle z^2 \frac{\partial^2 V}{\partial z^2} \right\rangle \right)$$

- no linear term ... no net $\vec{E}$ field or "direction" to electron blob.
- no direction special ...

$$\left\langle x^2 \frac{\partial^2 V}{\partial x^2} \right\rangle + \left\langle y^2 \frac{\partial^2 V}{\partial y^2} \right\rangle + \left\langle z^2 \frac{\partial^2 V}{\partial z^2} \right\rangle$$

$$\simeq \frac{1}{3} \langle r^2 \rangle \underbrace{\langle \nabla^2 V \rangle}_{-\frac{e\rho(\vec{r})}{\varepsilon_0}} \quad \text{poisson equation}$$

$\rho(\vec{x}) = e\,\delta(\vec{r})$ proton

so

$$\langle V_e(\vec{r}) \rangle \simeq -\frac{e^2}{4\pi\varepsilon_0} \frac{1}{r} + \boxed{\frac{e^2}{6\varepsilon_0} \underbrace{\langle r^2 \rangle}_{= \lambda\!\!\!^{-2} = \alpha^2 a^2} \delta(\vec{r})}$$

only influences when electron sits on proton

Darwin Term

The perturbation introduced by the Darwin term only influences $l=0$ states, because they are the only ones where the electron sits on the positron. The electron energy gets pushed <u>up</u>. By how much (approximately)?

$$\langle H_{Darwin}\rangle = \frac{e^2}{6\varepsilon_0}\alpha^2 a^2 \underbrace{\langle \psi_{n00}|\,\delta(\vec{r})\,|\psi_{n00}\rangle}_{=|\psi_{n00}(\vec{0})|^2}$$

p. 151, Eq. 4.80 $a \to a\cdot n$.

$$|\psi_{n00}(0)|^2 = \frac{1}{\pi a^3 n^3}$$

(p 150, Eq. 4.73)

$$\boxed{\langle H_D\rangle = \frac{1}{6}\alpha^2 \frac{e^2}{4\pi\varepsilon_0 a}\frac{1}{n^4}\cdot 4n}$$

($l=0$ only).

note:

$$\frac{E_n^2}{mc^2} = \underbrace{\frac{m^2}{4\hbar^4}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^4\frac{1}{n^4}}_{\text{p.149, Eq 4.70}}\cdot\frac{1}{mc^2}$$

$$= \frac{1}{4}\underbrace{\left(\frac{e^2}{4\pi\varepsilon_0\hbar c}\right)^2}_{\alpha^2}\cdot\left(\frac{e^2}{4\pi\varepsilon_0}\right)\underbrace{\left(\frac{me^2}{\hbar^2 4\pi\varepsilon_0}\right)}_{\frac{1}{a}}\frac{1}{n^4}$$

$$\frac{E_n^2}{mc^2} = \frac{1}{4}\alpha^2\frac{e^2}{4\pi\varepsilon_0 a}\frac{1}{n^4}$$

go back and compare with the difference between (α) + (β), which is

$$\boxed{+\frac{1}{2}\frac{E_n^2}{mc^2}\cdot 4n = \frac{1}{8}\alpha^2\frac{e^2}{4\pi\varepsilon_0 a}\frac{1}{n^4}\cdot 4n}$$

Compare, "almost same"!

After 3 pages of work ...

$$E_D^1 = \alpha^2 |E_n| \underbrace{\left[+\frac{1}{n^2} \cdot n \right]}_{\varepsilon_D^1(n,0)} \quad \text{when } l = 0$$

$$= 0 \qquad l > 0$$

Add this to the table!

$\varepsilon_D^1(n,l)$

	$l=0$	$l=1$	$l=2$	$l=3$	$l=4$
$n=5$	-0.170	-0.103	-0.030	-0.0271	-0.0144
$n=4$	-0.203	-0.120	-0.053	-0.0246	
$n=3$	-0.250	-0.139	-0.050		
$n=2$	-0.313	-0.146			
$n=1$	-0.250				

revised

$$\varepsilon_{rd} = \varepsilon_r^1(n,0) + \varepsilon_D^1(n,0) = \left[-\frac{1}{n^2}\left[n - \frac{3}{4} \right] \right] \quad l = 0$$

$$= \varepsilon_r^1(n,l) = \left[-\frac{1}{n^2}\left[\frac{n}{l+\frac{1}{2}} - \frac{3}{4} \right] \right] \quad l > 0$$

(c) Spin-Orbit

$$E_{so}^1 = 0 \qquad l = 0$$

$$= \frac{\alpha^2}{n^2} |E_n| \left(\frac{n\left[j(j+1) - l(l+1) - 3/4 \right]}{2l(l+1/2)(l+1)} \right)$$

Two cases:

(i) $j_- = l - \frac{1}{2}$ or $l = j_- + \frac{1}{2}$

then $$E^1_{so-} = \frac{\alpha^2}{n^2}|E_n|\left(\frac{n[j_-(j_-+1) - (j_-+\frac{1}{2})(j_-+\frac{3}{2}) - 3/4]}{2(j_-+\frac{1}{2})(j_-+1)(j_-+\frac{3}{2})}\right)$$

$$= \frac{\alpha^2}{n^2}|E_n|\left(\frac{n[\cancel{j_-^2}+j_- - \cancel{j_-^2} - 2j_- - \frac{3}{4} - \frac{3}{4}]}{2(j_-+\frac{1}{2})(j_-+1)(j_-+3/2)}\right)$$

$$= \frac{\alpha^2}{n^2}|E_n|\left(\frac{n(-j_- - 3/2)}{2(j_-+\frac{1}{2})(j_-+1)(j_-+3/2)}\right)$$

$$\boxed{E^1_{so-} = -\frac{\alpha^2}{n^2}|E_n|\left(\frac{n}{(2j_-+1)(j_-+1)}\right) \qquad (j_- = l - \tfrac{1}{2})}$$

(ii) $j_+ = l + \frac{1}{2}$ or $l = j_+ - \frac{1}{2}$

$$E^1_{so+} = \frac{\alpha^2}{n^2}|E_n|\left(\frac{n[j_+(j_++1) - (j_+-\frac{1}{2})(j_++\frac{1}{2}) - 3/4]}{2(j_+-\frac{1}{2})\,j_+\,(j_++\frac{1}{2})}\right)$$

$$= \frac{\alpha^2}{n^2}|E_n|\left(\frac{n[\cancel{j_+^2}+j_+ - \cancel{j_+^2} + \frac{1}{4} - \frac{3}{4}]}{2(j_+-\frac{1}{2})\,j_+\,(j_++\frac{1}{2})}\right)$$

$$= \frac{\alpha^2}{n^2}|E_n|\left(\frac{n[\cancel{j_+-\frac{1}{2}}]}{2\cancel{(j_+-\frac{1}{2})}\,j_+\,(j_++\frac{1}{2})}\right)$$

$$\boxed{E^1_{so+} = +\frac{\alpha^2}{n^2}|E_n|\left(\frac{n}{(2j_++1)j_+}\right) \qquad j_+ = l + \tfrac{1}{2}}$$

Note: $(2j_-+1)E^1_{so-} + (2j_++1)E^1_{so+} = 0$

"center of gravity" unchanged since $j_+ = j_- + 1$

Now include, $l > 0$

$$\varepsilon'_{so}(n, j_-) = -\left(\frac{1}{n}\,\frac{1}{(2j_-+1)(j_-+1)}\right) \qquad j_- = l - \tfrac{1}{2}$$

$$\varepsilon'_{tot}(n, l, j_-) = \varepsilon'_r(n, l) + \varepsilon'_{so}(n, j_-)$$

$$\varepsilon'_{so}(n, j_+) = +\left(\frac{1}{n}\,\frac{1}{(2j_++1)\,j_+}\right)$$

$$\varepsilon'_{tot}(n, l, j_+) = \varepsilon'_r(n, l) + \varepsilon'_{so}(n, j_+)$$

Put this in a table, with care to <u>split</u> the $l > 0$ entries into two, j_- & j_+ :

		j		j		j				
n=5	-0.170	1/2	-0.070	3/2	-0.0367	5/2	-0.020	7/2	-0.010	9/2
			-0.170	1/2	-0.070	3/2	-0.0367	5/2	-0.020	7/2
n=4	-0.203	1/2	-0.0781	3/2	-0.0365	5/2	-0.0156	7/2		
			-0.203	1/2	-0.0781	3/2	-0.0365	5/2		
n=3	-0.250	1/2	-0.0833	3/2	-0.0278	5/2				
			-0.250	1/2	-0.0833	3/2				
n=2	-0.313	1/2	-0.0625	3/2						
			-0.313	1/2						
n=1	-0.250	1/2								
	$l=0$		$l=1$		$l=2$		$l=3$		$l=4$	

After all this,

$l=0$: $O(p^4)$ + Darwin

$l>0$: $O(p^4)$ + Spin-Orbit

the total shift depends only on j!!! (not which l is the "parent").

True only to $O(\alpha^4)$

Fascinating effect at $O(\alpha^5)$ . . .
"LAMB SHIFT", splits the states with $=j$ but different "parents"

$\Rightarrow$ NOBEL 1955

$\Rightarrow$ Willis Lamb ... went to LA High School.