# Homework 3 – 6.23, 24, 32, 33

## 6.23

The 8 states are, using the notation $|n\,l\,m_l\,m_s\rangle$

$$\left.|2\,0\,0\,\pm 1/2\rangle\right\}2$$
$$\left.|2\,1\,0\,\pm 1/2\rangle\right\}2$$
$$\left.|2\,1\,\pm 1\,\pm 1/2\rangle\right\}4$$

Since $n=2$ for all these states, the total energy is

$$E_{bohr} + E_{fs} + E_{Z,\,strong} = E_{tot}.$$

Adding eq 6.79 and 6.82,

$$E_{tot} = \frac{-13.6\,eV}{n^2}\left\{1 - \frac{\alpha^2}{n}\left[\frac{3}{4n} - \frac{l(l+1) - m_l m_s}{l(l+1/2)(l+1)}\right]\right\}$$

$$+ \mu_B B_{ext}(m_l + m_s)$$

Chugging through the math,

| State | $m_l + 2m_s$ | Energy |
|---|---|---|
| $\|2\,0\,0\,\pm 1/2\rangle$ | $\pm 1$ | $-3.4\,eV(1+5/16\,\alpha^2) \pm \mu_B B_{ext}$ |
| $\|2\,1\,0\,\pm 1/2\rangle$ | $\pm 1$ | $-3.4\,eV(1+7/48\,\alpha^2) \pm \mu_B B_{ext}$ |
| $\|2\,1\,1\,\pm 1/2\rangle$ | $2/0$ | $-3.4\,eV(1+1/16\,\alpha^2) + 2\mu_B B_{ext}$ |

6.23 con't

| State | $m_l + 2m_s$ | $E_{tot}$ |
|---|---|---|
| $\vert 2\ 1\ -1\ 1/2\rangle$ | 0 | $-3.4\,eV\left(1+\frac{11}{48}\alpha^2\right)$ |
| $\vert 2\ 1\ -1\ -1/2\rangle$ | $-2$ | $-3.4\,eV\left(1+ 1/16\,\alpha^2\right) - 2\mu_B B_{ext}$ |

If we ignore fine structure, There are 5 distinct states

$$\left( E_Z^1 = \underset{d=2}{0},\ \underset{d=2\text{ each}}{\pm \mu_B B_{ext}},\ \underset{d=1\text{ each}}{\pm 2\mu_B B_{ext}} \right)$$

6.24 When $l=0$, $j=s$ and the good quantum states are the same for both strong and weak fields.

So They want us to show That

$$E^1_{weak} = E^1_{fs} + E^1_{Z,weak} = E^1_{fs,strong} + E^1_{Z,strong}.$$

for $l=0$, $s=1/2$

$$E^1_{weak} = \frac{13.6\,\alpha^2 eV}{n^4}\left(\frac{3}{4} - \frac{n}{\cancel{l}\,\cancel{2}+1/2}\right) + \mu_B g_j B_{ext} m_j$$

eq 6.67 eq 6.74

$m_j = m_s$, $g_j = 2$ and

$$\boxed{E^1_{weak} = \frac{13.6\,\alpha^2 eV}{n^4}\left(\frac{3}{4} - n\right) + 2\mu_B B_{ext} m_s}$$

## 6.24 con't

$$E^1_{strong} = \overset{\text{eq 6.82}}{\frac{13.6\,eV}{n^4}\alpha^2\left(\frac{3}{4} - n\right)} + \overset{\text{eq 6.79}}{2\mu_B B_{ext} m_s}$$

so $E^1_{weak} = E^1_{strong}$. ∎
when $\ell = 0$.

## 6.32 Feynman-Hellmann

Here's one way to prove it:

$$E_n|\psi_n\rangle = H|\psi_n\rangle \qquad \text{(TISE)}$$

$$E_n \cancelto{1}{\langle\psi_n|\psi_n\rangle} = \langle\psi_n|H|\psi_n\rangle$$

$$\frac{\partial E_n}{\partial\lambda} = \frac{\partial}{\partial\lambda}\langle\psi_n|H|\psi_n\rangle$$

$$= \langle\frac{\partial\psi_n}{\partial\lambda}|H|\psi_n\rangle + \langle\psi_n|\frac{\partial H}{\partial\lambda}|\psi_n\rangle + \langle\psi_n|H|\frac{\partial\psi_n}{\partial\lambda}\rangle$$

$$= \langle\psi_n|\frac{\partial H}{\partial\lambda}|\psi_n\rangle + E_n\left(\langle\frac{\partial\psi_n}{\partial\lambda}|\psi\rangle + \langle\psi|\frac{\partial\psi}{\partial\lambda}\rangle\right)$$

$$= \langle\psi_n|\frac{\partial H}{\partial\lambda}|\psi_n\rangle + E_n\frac{\partial}{\partial\lambda}\cancelto{1}{\langle\psi_n|\psi_n\rangle}$$

$$= \langle\psi_n|\frac{\partial H}{\partial\lambda}|\psi_n\rangle. \quad ∎$$

6.32 con't

$$\frac{\partial E_n}{\partial \lambda} = \langle \psi_n | \frac{\partial H}{\partial \lambda} | \psi_n \rangle$$

(i) $$\frac{\partial E_n}{\partial \omega} = \frac{\partial}{\partial \omega} \hbar\omega(n+\tfrac{1}{2}) = \hbar(n+\tfrac{1}{2}) = \langle \psi_n | \frac{\partial}{\partial \omega}\left(\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 x^2\right) | \psi_n \rangle$$

$$\hbar(n+\tfrac{1}{2}) = \langle \psi_n | m\omega x^2 | \psi_n \rangle$$

so $$\langle x^2 \rangle \equiv \langle \psi_n | x^2 | \psi_n \rangle = \frac{\hbar}{m\omega}(n+\tfrac{1}{2})$$

$$\underline{\langle V \rangle = \frac{1}{2}m\omega^2 \langle x^2 \rangle = \frac{\hbar\omega}{2}(n+\tfrac{1}{2})}$$

(ii) $$\frac{\partial E_n}{\partial \hbar} = \omega(n+\tfrac{1}{2}) = \langle \psi_n | \frac{\partial}{\partial \hbar}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2\right) | \psi_n \rangle$$

$$= \langle \psi_n | -\frac{\hbar}{m}\frac{\partial^2}{\partial x^2} | \psi_n \rangle$$

$$= \langle \psi_n | \frac{2T}{\hbar} | \psi_n \rangle$$

so $$\underline{\langle T \rangle = \frac{\hbar\omega}{2}(n+\tfrac{1}{2})}$$

## 6.32 con't

(iii)

$$\frac{\partial E_n}{\partial m} = 0 = \langle \psi_n | \frac{\partial}{\partial M}\left( \frac{-\hbar^2}{2M}\frac{\partial^2}{\partial x^2} + \frac{1}{2} m\omega^2 x^2 \right) | \psi_n \rangle$$

$$= \langle \psi_n | \frac{\hbar^2}{2m^2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\omega^2 x^2 | \psi_n \rangle .$$

$$= \left\langle -\frac{T}{m} + \frac{V}{m} \right\rangle = \frac{1}{m}\left( \langle V \rangle - \langle T \rangle \right)$$

so $\langle V \rangle = \langle T \rangle$ .

Therefore the Virial Theorem results of 2.12 check out.

## 6.33

$$H = -\frac{\hbar^2}{2M}\frac{d^2}{dr^2} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2} - \frac{e^2}{4\pi\epsilon_0}\frac{1}{r}$$

$$E_n = -\frac{me^4}{32\pi^2\epsilon_0^2\hbar^2 \underbrace{(j_{max}+1+l)^2}_{n^2 \text{ by eq } 4.67}}$$

$$\frac{\partial E_n}{\partial e} = \frac{-4me^3}{32\pi^2\epsilon_0^2\hbar^2 n^2} = \langle \psi_n | -\frac{e}{2\pi\epsilon_0}\frac{1}{r} | \psi_n \rangle ,$$

$$\left\langle \frac{1}{r} \right\rangle = \frac{4me^2}{16\pi^2\epsilon_0\hbar^2 n^2} = \frac{me^2}{4\pi\epsilon_0\hbar^2 n^2} = \frac{1}{n^2 a} .$$

(b) $$\frac{\partial E_n}{\partial l} = \frac{me^4}{16\pi^2 \epsilon_0^2 \hbar^2 \underbrace{(j_{max}+l+1)^3}_{n^3}}$$

$$= \langle \psi_n | \frac{\hbar^2}{2mr^2}(2l+1) | \psi_n \rangle$$

$$= \langle \frac{1}{r^2} \rangle \left( \frac{2m}{(2l+1)\hbar^2} \right)^{-1}$$

So $$\langle \frac{1}{r^2} \rangle = \frac{2m^2 e^4}{(2l+1)\hbar^4 \, 16\pi^2 n^2 \epsilon_0^2}$$

$$= \frac{1}{n^3 (l+\frac{1}{2}) a}$$

Good!