The 3x 3 matrices $\underset{\sim}{J}_i^{(1)}$

easy: $i=3$ aka $\underset{\sim}{J}_z$: $\langle 1m'|\underset{\sim}{J}_z|1m\rangle$

$$= m\hbar\, \delta_{m',m}$$

$$\begin{array}{c|ccc} & m=1 & 0 & -1 \\ \hline m'=1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{array}$$

$\underset{\sim}{J}_+$: $\langle 1m'|\underset{\sim}{J}_+|1m\rangle = \langle 1m'|\hbar\sqrt{1(1+1)-m(m+1)}\,|1\,m+1\rangle$

$$= \hbar\sqrt{2-m(m+1)}\,\delta_{m',m+1}$$

$$\begin{array}{c|ccc} & m=1 & 0 & -1 \\ \hline m'=1 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{2} \\ -1 & 0 & 0 & 0 \end{array}\ \hbar \doteq \underset{\sim}{J}_+$$

$\underset{\sim}{J}_- = \underset{\sim}{J}_+^\dagger$

$$\underset{\sim}{J}_- \doteq \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}\hbar$$

$$\underset{\sim}{J}_x = \frac{1}{2}(\underset{\sim}{J}_+ + \underset{\sim}{J}_-) \doteq \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\underset{\sim}{J}_y = \frac{1}{2i}(\underset{\sim}{J}_+ - \underset{\sim}{J}_-) \doteq \frac{\hbar}{2i}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

note: $\underset{\sim}{S}_3 \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ $\quad \underset{\sim}{S}_3^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq \underset{\sim}{\mathbb{1}}$

$$\underset{\sim}{\sigma}_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \underset{\sim}{\sigma}_3^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underset{\sim}{\mathbb{1}}$$

but do note: $\underset{\sim}{S}_3^3 = \underset{\sim}{S}_3$

as $\underset{\sim}{\sigma}_1^3 = \underset{\sim}{\sigma}_1$

and $\underset{\sim}{S}_1 = \begin{pmatrix} 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \end{pmatrix}$ $\quad \underset{\sim}{S}_1^2 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$

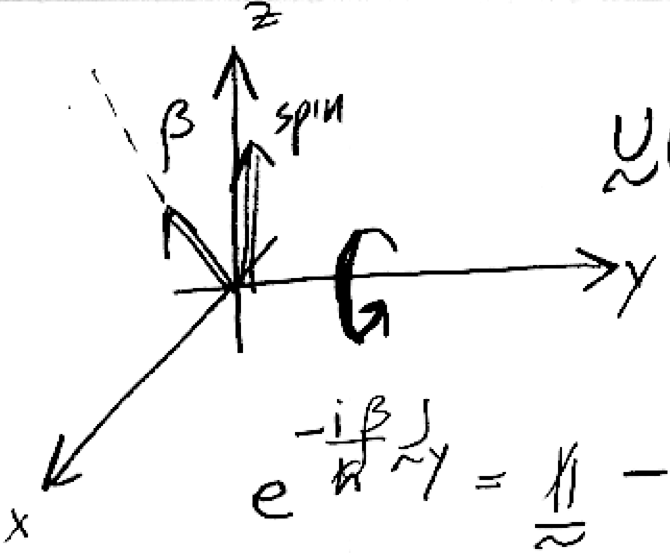
$$\underset{\sim}{S}_1^3 = \begin{pmatrix} 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \end{pmatrix} = \underset{\sim}{S}_1$$

and $\underset{\sim}{S}_2 = \begin{pmatrix} 0 & -i/\sqrt{2} & 0 \\ i/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & i/\sqrt{2} & 0 \end{pmatrix}$ $\quad \underset{\sim}{S}_2^2 = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$

$$\underset{\sim}{S}_3^3 = \begin{pmatrix} 0 & -i/\sqrt{2} & 0 \\ i/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & i/\sqrt{2} & 0 \end{pmatrix} = \underset{\sim}{S}_2$$

These relationships come in useful when evaluating matrices that describe rotations.

$\Rightarrow$ interesting case is that that describes rotations about $\underline{y}$ axis:



$$\underset{\sim}{U}(R[\beta\hat{j}]) = e^{-\frac{i\beta}{\hbar}\underset{\sim}{J}_y}$$

$$e^{-\frac{i\beta}{\hbar}\underset{\sim}{J}_y} = \underset{\sim}{\mathbb{1}} - \frac{i\beta}{\hbar}\underset{\sim}{J}_y + \frac{1}{2!}\left(\frac{-i\beta}{\hbar}\right)^2\underset{\sim}{J}_y^2 + \frac{1}{3!}\left(\frac{-i\beta}{\hbar}\right)^3\underset{\sim}{J}_y^3 + \frac{1}{4!}\left(\frac{-i\beta}{\hbar}\right)^4\underset{\sim}{J}_z^4$$

$$\boxed{\text{when } j = \frac{1}{2}}$$

$$\underset{\sim}{J}_y = \frac{\hbar}{2}\underset{\sim}{\sigma}_2 \qquad \underset{\sim}{J}_y^2 = \left(\frac{\hbar}{2}\right)^2\underset{\sim}{\sigma}_2^2 = \left(\frac{\hbar}{2}\right)\cdot\underset{\sim}{\mathbb{1}}$$

$$\underset{\sim}{J}_y^3 = \left(\frac{\hbar}{2}\right)^3\underset{\sim}{\sigma}_2 \qquad \underset{\sim}{J}_y^4 = \left(\frac{\hbar}{2}\right)^4\cdot\underset{\sim}{\mathbb{1}}$$

$$e^{-\frac{i\beta}{\hbar}\underset{\sim}{J}_y} \doteq \underset{\sim}{\mathbb{1}} - \frac{i\beta}{2}\underset{\sim}{\sigma}_2 + \frac{1}{2!}\left(-\frac{i\beta}{2}\right)^2\cdot\underset{\sim}{\mathbb{1}} + \frac{1}{3!}\left(-\frac{i\beta}{2}\right)^3\underset{\sim}{\sigma}_2 + \frac{1}{4!}\left(-\frac{i\beta}{2}\right)^4\cdot\underset{\sim}{\mathbb{1}}$$

$$\doteq \underset{\sim}{\mathbb{1}}\cdot\cos\frac{\beta}{2} - \underset{\sim}{\sigma}_2\sin\frac{\beta}{2}$$

$$\doteq \begin{pmatrix} \cos\frac{\beta}{2} & i\sin\frac{\beta}{2} \\ -i\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}$$

when $j = 1$

$\swarrow$ since $\hbar \underset{\sim}{S}_2$ is rep of $\underset{\sim}{J}_y$

$$e^{-\frac{i\beta}{\hbar}\underset{\sim}{J}_y} \doteq \underset{\sim}{\mathbb{1}} - i\beta \underset{\sim}{S}_2 + \frac{1}{2!}(-i\beta)^2 \underset{\sim}{S}_2^{\,2}$$

$$+ \frac{1}{3!}(-i\beta)^3 \underset{\sim}{S}_2 + \frac{1}{4!}(-i\beta)^4 \underset{\sim}{S}_2^{\,2}$$

$$\doteq \underset{\sim}{\mathbb{1}} - \underset{\sim}{S}_2 \sin\beta + \underset{\sim}{S}_2^{\,2}(\cos\beta - 1)$$

$$\uparrow \qquad\qquad\qquad \nwarrow$$

$$\begin{pmatrix} 0 & -i/\sqrt{2} & 0 \\ i/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & i/\sqrt{2} & 0 \end{pmatrix} \qquad \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$\doteq \begin{pmatrix} \frac{1}{2}(1+\cos\beta) & i/\sqrt{2}\,\sin\beta & \frac{1}{2}(1-\cos\beta) \\ -i/\sqrt{2}\,\sin\beta & \cos\beta & i/\sqrt{2}\,\sin\beta \\ \frac{1}{2}(1-\cos\beta) & -i/\sqrt{2}\,\sin\beta & \frac{1}{2}(1+\cos\beta) \end{pmatrix}$$

12.5.3

(1) Show $\langle \underset{\sim}{J}_x \rangle = \langle \underset{\sim}{J}_y \rangle = 0$ in $|jm\rangle$

$$\langle jm| \underset{\sim}{J}_x |jm\rangle = \frac{1}{2}\langle jm|(\underbrace{\underset{\sim}{J}_+ + \underset{\sim}{J}_-}_{\substack{\uparrow \quad \uparrow \\ \text{raises} \quad \text{lowers}}})|jm\rangle$$

in either case, orthogonal to the bra $\langle jm|$

$$= 0.$$

$$\langle jm| \underset{\sim}{J}_y |jm\rangle = \frac{1}{2i}\underbrace{\langle jm|(\underset{\sim}{J}_+ - \underset{\sim}{J}_-)|j}_{0 \text{ by same argument}} m\rangle$$

(2) Show $\langle \underset{\sim}{J}_x^2 \rangle = \langle \underset{\sim}{J}_y^2 \rangle = \frac{1}{2}\hbar^2[j(j+1) - m^2]$.

$$\underset{\sim}{J}_x^2 = \frac{1}{2}(\underset{\sim}{J}_+ + \underset{\sim}{J}_-)\cdot\frac{1}{2}(\underset{\sim}{J}_+ + \underset{\sim}{J}_-)$$

$$= \frac{1}{4}(\underset{\sim}{J}_+^2 + \underset{\sim}{J}_+\underset{\sim}{J}_- + \underset{\sim}{J}_-\underset{\sim}{J}_+ + \underset{\sim}{J}_-^2)$$

$$\langle \underset{\sim}{J}_x^2 \rangle = \frac{1}{4}(\underbrace{\cancel{\langle \underset{\sim}{J}_+^2 \rangle}}_{0} + \underbrace{\langle \underset{\sim}{J}_+\underset{\sim}{J}_- + \underset{\sim}{J}_-\underset{\sim}{J}_+ \rangle}_{\text{this is non-zero}} + \underbrace{\cancel{\langle \underset{\sim}{J}_-^2 \rangle}}_{0})$$

$$\langle \underset{\sim}{J}_x^2 \rangle = \frac{1}{4}\langle \underset{\sim}{J}_+\underset{\sim}{J}_- + \underset{\sim}{J}_-\underset{\sim}{J}_+ \rangle$$

$$\underset{\sim}{J}_y^2 = \frac{1}{2i}(\underset{\sim}{J}_+ - \underset{\sim}{J}_-)\frac{1}{2i}(\underset{\sim}{J}_+ - \underset{\sim}{J}_-)$$

$$= -\frac{1}{4}(\underset{\sim}{J}_+^2 - \underset{\sim}{J}_+\underset{\sim}{J}_- - \underset{\sim}{J}_-\underset{\sim}{J}_+ + \underset{\sim}{J}_-^2)$$

$$\langle \underset{\sim}{J}_y^2 \rangle = \frac{1}{4}\left(-\langle \underset{\sim}{J}_+^2 \rangle + \langle \underset{\sim}{J}_+ \underset{\sim}{J}_- + \underset{\sim}{J}_- \underset{\sim}{J}_+ \rangle - \langle \underset{\sim}{J}_-^2 \rangle\right)$$

$$\langle \underset{\sim}{J}_y^2 \rangle = \frac{1}{4} \langle \underset{\sim}{J}_+ \underset{\sim}{J}_- + \underset{\sim}{J}_- \underset{\sim}{J}_+ \rangle = \langle \underset{\sim}{J}_x^2 \rangle$$

$\langle jm | \underset{\sim}{J}_- \underset{\sim}{J}_+ | jm \rangle$ is norm of $\underset{\sim}{J}_+ | jm \rangle$
since adjoint is $\langle jm | \underset{\sim}{J}_-$

$$\langle jm | \underset{\sim}{J}_- \underset{\sim}{J}_+ | jm \rangle = \hbar^2 (j(j+1) - m(m+1))$$

$\langle jm | \underset{\sim}{J}_+ \underset{\sim}{J}_- | jm \rangle$ is norm of $\underset{\sim}{J}_- | jm \rangle$

$$= \hbar^2 (j(j+1) - m(m-1))$$

$$\langle \underset{\sim}{J}_x^2 \rangle = \langle \underset{\sim}{J}_y^2 \rangle = \frac{1}{4}\left[\hbar^2(j(j+1) - m(m+1)) + \hbar^2(j(j+1) - m(m-1))\right]$$

$$\boxed{\langle \underset{\sim}{J}_x^2 \rangle = \langle \underset{\sim}{J}_y^2 \rangle = \frac{1}{2}\hbar^2 \left[j(j+1) - m^2\right]}$$

(3) $(\Delta J_x)^2 = \langle \hat{\underset{\sim}{J}}_x^2 \rangle \quad = \langle \underset{\sim}{J}_x^2 \rangle$

$$\hat{\underset{\sim}{J}}_x = \underset{\sim}{J}_x - \underset{0}{\cancel{\langle \underset{\sim}{J}_x \rangle}} = \underset{\sim}{J}_x$$

$$(\Delta J_x)^2 = \langle \underset{\sim}{J}_x^2 \rangle = \frac{1}{2}\hbar^2 \left[j(j+1) - m^2\right]$$

$$(\Delta J_y)^2 = \langle \hat{\underset{\sim}{J}}_y^2 \rangle$$

$$\hat{\underset{\sim}{J}}_y = \underset{\sim}{J}_y - \underset{0}{\cancel{\langle \underset{\sim}{J}_y \rangle}} = \underset{\sim}{J}_y$$

$$(\Delta J_y)^2 = \langle \underset{\sim}{J}_y^2 \rangle = \frac{1}{2}\hbar^2 \left[j(j+1) - m^2\right]$$

so

$$(\Delta J_x)^2(\Delta J_y)^2 = \frac{1}{4}\hbar^4\left[j(j+1)-m^2\right]^2$$

$$\Delta J_x \Delta J_y = \frac{1}{2}\hbar\left[j(j+1)-m^2\right]$$

Uncertainty Principle

$$(\Delta J_x)^2(\Delta J_y)^2 \geq \frac{1}{4}\left|\left\langle[\hat{\underset{\sim}{J}}_x,\hat{\underset{\sim}{J}}_y]_+\right\rangle\right|^2 + \frac{1}{4}\left|\left\langle\underbrace{\underbrace{[\underset{\sim}{J}_x,\underset{\sim}{J}_y]}_{i\hbar\underset{\sim}{J}_z}}_{i\hbar m}\right\rangle\right|^2$$

$\uparrow \quad \uparrow$

since $\langle\underset{\sim}{J}_x\rangle = \langle\underset{\sim}{J}_y\rangle = 0$

$$[\hat{\underset{\sim}{J}}_x,\hat{\underset{\sim}{J}}_y]_+ = [\underset{\sim}{J}_x,\underset{\sim}{J}_y]_+$$

$$[\underset{\sim}{J}_x,\underset{\sim}{J}_y]_+ = \underset{\sim}{J}_x\underset{\sim}{J}_y + \underset{\sim}{J}_y\underset{\sim}{J}_x$$

$$= \frac{1}{4i}\left((\underset{\sim}{J}_+ + \underset{\sim}{J}_-)(\underset{\sim}{J}_+ - \underset{\sim}{J}_-) + (\underset{\sim}{J}_+ - \underset{\sim}{J}_-)(\underset{\sim}{J}_+ + \underset{\sim}{J}_-)\right)$$

$$= \frac{1}{4i}\left(2\underset{\sim}{J}_+^2 - 2\underset{\sim}{J}_-^2 - \underset{\sim}{J}_+\underset{\sim}{J}_- + \underset{\sim}{J}_-\underset{\sim}{J}_+ + \underset{\sim}{J}_+\underset{\sim}{J}_- - \underset{\sim}{J}_-\underset{\sim}{J}_+\right)$$

$0 \qquad 0$

$$[\underset{\sim}{J}_x,\underset{\sim}{J}_y]_+ = \frac{1}{4i}\left(2\underset{\sim}{J}_+^2 - 2\underset{\sim}{J}_-^2\right) = \frac{1}{2i}\left(\underset{\sim}{J}_+^2 - \underset{\sim}{J}_-^2\right)$$

$$\left\langle[\underset{\sim}{J}_x,\underset{\sim}{J}_y]_+\right\rangle = \frac{1}{2i}\left\langle(\underset{\sim}{J}_+^2 - \underset{\sim}{J}_-^2)\right\rangle = 0\,!$$

so,

$$(\Delta J_x)^2(\Delta J_y)^2 \geq 0 + \frac{1}{4}\hbar^2 m^2$$

or

$$\Delta J_x \Delta J_y \geq \frac{1}{2}\hbar m$$

or $\cancel{\frac{1}{2}\hbar}[j(j+1)-m^2] \geq \cancel{\frac{1}{2}\hbar}|m|$

$$j(j+1) \geq m^2 + |m|$$

which is indeed true since

$$-j \leq m \leq j$$

(4) when $m=j$

$j(j+1) \geq j(j+1)$ and the equality holds

when $m=-j$

$j(j+1) \geq j^2 + |-j| = j(j+1)$

and again, the equality holds

Extra Credit: equality above implies

$$\hat{\underset{\sim}{J}}_x |j \pm j\rangle = \lambda \hat{\underset{\sim}{J}}_y |j \pm j\rangle$$

Check:

$$\frac{1}{2}(\underset{\sim}{J}_+ + \underset{\sim}{J}_-)|j \pm j\rangle \overset{?}{=} \lambda \frac{1}{2i}(\underset{\sim}{J}_+ - \underset{\sim}{J}_-)|j \pm j\rangle$$

$m=+j$: $\underset{\sim}{J}_+$ wipes out, and $\lambda = -i$ from $\underset{\sim}{J}_-$ alone

$m=-j$, $\underset{\sim}{J}_-$ wipes out, and $\lambda = i$ from $\underset{\sim}{J}_+$ alone.