These are useful because:

$$\underset{\sim}{J}_x = \frac{1}{2}(\underset{\sim}{J}_+ + \underset{\sim}{J}_-) \qquad \underset{\sim}{J}_y = \frac{1}{2i}(\underset{\sim}{J}_+ - \underset{\sim}{J}_-)$$

$j=0$ is trivial... all matrices 0.

$j=\frac{1}{2}$ first interesting one...

$$\langle \tfrac{1}{2} m' | \vec{\underset{\sim}{J}}^2 | \tfrac{1}{2} m \rangle = \delta_{m'm} \times \tfrac{1}{2}(\tfrac{1}{2}+1)\hbar^2 = \tfrac{3}{4}\hbar^2 \delta_{m'm}$$

$$\doteq \tfrac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \tfrac{3}{4}\hbar^2 \cdot \underset{\sim}{\mathbb{1}}$$

$$\langle \tfrac{1}{2} m' | \underset{\sim}{J}_z | \tfrac{1}{2} m \rangle = \delta_{m'm} \times m \times \hbar \qquad \begin{pmatrix} m \text{ can be} \\ \pm\frac{1}{2} \end{pmatrix}$$

$$\doteq \tfrac{1}{2}\hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \tfrac{1}{2}\hbar \underset{\sim}{\sigma}_z \leftarrow \tfrac{1}{2}\hbar \underset{\sim}{\sigma}_3$$

$$\langle \tfrac{1}{2} m' | \underset{\sim}{J}_+ | \tfrac{1}{2} m \rangle = \underbrace{\hbar\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) - m(m+1)}}_{\substack{0 \text{ when } m=\frac{1}{2} \\ \neq 0 \text{ when } m=-\frac{1}{2},\ m'=\frac{1}{2} \text{ only}}} \delta_{m'\, m+1}$$

$$\text{then} = \hbar\sqrt{\tfrac{3}{4} + \tfrac{1}{2}(\tfrac{1}{2})} = \hbar$$

$$\doteq \hbar \begin{array}{c|cc} & m=\frac{1}{2} & -\frac{1}{2} \\ \hline m'=\frac{1}{2} & 0 & 1 \\ -\frac{1}{2} & 0 & 0 \end{array} = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\langle \tfrac{1}{2} m' | \underset{\sim}{J}_- | \tfrac{1}{2} m \rangle = \hbar\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) - m(m-1)}\ \delta_{m'\, m-1}$$

$$\doteq \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

so $\langle \tfrac{1}{2} m' | \underset{\sim}{J}_x | \tfrac{1}{2} m \rangle = \tfrac{1}{2} \langle \tfrac{1}{2} m' | (\underset{\sim}{J}_+ + \underset{\sim}{J}_-) | \tfrac{1}{2} m \rangle$

$$\doteq \tfrac{1}{2}\hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \tfrac{1}{2}\hbar \underset{\sim}{\sigma}_x = \tfrac{1}{2}\hbar \underset{\sim}{\sigma}_1$$

$$\langle \tfrac{1}{2} m' | \underset{\sim}{J}_y | \tfrac{1}{2} m \rangle = \tfrac{1}{2i} \langle \tfrac{1}{2} m' | (\underset{\sim}{J}_+ - \underset{\sim}{J}_-) | \tfrac{1}{2} m \rangle$$

$$\doteq \frac{\hbar}{2i} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \tfrac{1}{2}\hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \tfrac{1}{2}\hbar \underset{\sim}{\sigma}_y = \tfrac{1}{2}\hbar \underset{\sim}{\sigma}_3$$

These are on page 328

Angular Momentum Eigenfunctions

eigenvalue of $\underset{\sim}{J}^2 = \underset{\sim}{L}^2$ is $l(l+1)\hbar^2$

l is full integer

use $\underset{\sim}{L}_+$, $\underset{\sim}{L}_-$

recall $\underset{\sim}{L}_z \doteq \frac{\hbar}{i} \frac{\partial}{\partial \phi}$

$$\underset{\sim}{L}_\pm = \underset{\sim}{L}_x \pm i \underset{\sim}{L}_y$$

how to represent $\underset{\sim}{L}_x$ and $\underset{\sim}{L}_y$?

$$\underset{\sim}{L}_x = \underset{\sim}{y} \underset{\sim}{p}_z - \underset{\sim}{z} \underset{\sim}{p}_y \doteq \frac{\hbar}{i}\left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right)$$

$$\frac{\partial}{\partial z} = \begin{pmatrix} \frac{\partial r}{\partial z} & \frac{\partial \theta}{\partial z} & \frac{\partial \phi}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \phi} \end{pmatrix}$$

$r = \sqrt{x^2+y^2+z^2}$ $\qquad \frac{\partial r}{\partial z} = \frac{\frac{1}{2} \cdot 2z}{\sqrt{x^2+y^2+z^2}} = \frac{z}{r} = \cos\theta$

$$\cos\theta = \frac{z}{r} \qquad -\sin\theta\frac{\partial\theta}{\partial z} = \frac{1}{r} - \frac{z^2}{r^3} = \frac{r^2-z^2}{r^3}$$

$$= \frac{x^2+y^2}{r^3} = \frac{\sin^2\theta}{r}$$

$$\frac{\partial\theta}{\partial z} = -\frac{\sin\theta}{r}$$

$$\tan\phi = \frac{y}{x} \rightarrow \frac{\partial\phi}{\partial z} = 0$$

$$\frac{\partial}{\partial z} = \begin{pmatrix}\cos\theta & -\frac{\sin\theta}{r} & 0\end{pmatrix}\begin{pmatrix}\frac{\partial}{\partial r}\\ \frac{\partial}{\partial\theta}\\ \frac{\partial}{\partial\phi}\end{pmatrix} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial y} = \begin{pmatrix}\frac{\partial r}{\partial y} & \frac{\partial\theta}{\partial y} & \frac{\partial\phi}{\partial y}\end{pmatrix}\begin{pmatrix}\frac{\partial}{\partial r}\\ \frac{\partial}{\partial\theta}\\ \frac{\partial}{\partial\phi}\end{pmatrix}$$

$$r = \sqrt{x^2+y^2+z^2} \qquad \frac{\partial r}{\partial y} = \frac{\frac{1}{2}2y}{\sqrt{x^2+y^2+z^2}} = \frac{y}{r} = \sin\theta\sin\phi$$

$$\cos\theta = \frac{z}{r} \qquad -\sin\theta\frac{\partial\theta}{\partial y} = \frac{-yz}{r^3} = \frac{-\sin\theta\cos\theta\sin\phi}{r}$$

$$\tan\phi = \frac{y}{x} \Rightarrow \frac{1}{\cos^2\phi}\frac{\partial\phi}{\partial y} = \frac{1}{x} \Rightarrow \frac{\partial\phi}{\partial y} = \frac{\cos^2\phi}{r\sin\theta\cos\phi} = \frac{\cos\phi}{r\sin\theta}$$

$$\frac{\partial}{\partial y} = \begin{pmatrix}\sin\theta\sin\phi & +\frac{\cos\theta\sin\phi}{r} & \frac{\cos\phi}{r\sin\theta}\end{pmatrix}\begin{pmatrix}\frac{\partial}{\partial r}\\ \frac{\partial}{\partial\theta}\\ \frac{\partial}{\partial\phi}\end{pmatrix}$$

$$= \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$\underset{\sim}{L}_x = \frac{\hbar}{i}\Bigg(\overbrace{r\sin\theta\sin\phi}^{y}\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)$$

$$- \underbrace{r\cos\theta}_{z}\left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\Bigg)$$

$$= \frac{\hbar}{i}\left(-(\sin^2\theta+\cos^2\theta)\sin\phi\frac{\partial}{\partial\theta} - \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$\boxed{\underset{\sim}{L}_x = \frac{\hbar}{i}\left(-\sin\phi\frac{\partial}{\partial\theta} - \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right)}$$

$$\underset{\sim}{L}_y = \underset{\sim}{z}p_x - \underset{\sim}{x}p_z \doteq z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial x} = \begin{pmatrix}\frac{\partial r}{\partial x} & \frac{\partial\theta}{\partial x} & \frac{\partial\phi}{\partial x}\end{pmatrix}\begin{pmatrix}\frac{\partial}{\partial r}\\ \frac{\partial}{\partial\theta}\\ \frac{\partial}{\partial\phi}\end{pmatrix}$$

$$r = \sqrt{x^2+y^2+z^2} \quad\Rightarrow\quad \frac{\partial r}{\partial x} = \frac{x}{r} = \sin\theta\cos\phi$$

$$\cos\theta = \frac{z}{r} \qquad -\sin\theta\frac{\partial\theta}{\partial x} = -\frac{zx}{r^3} = -\frac{\sin\theta\cos\theta\cos\phi}{r}$$

$$\frac{\partial\theta}{\partial x} = \frac{\cos\theta\cos\phi}{r}$$

$$\tan\phi = \frac{y}{x} \qquad \frac{1}{\cos^2\phi}\frac{\partial\phi}{\partial x} = \frac{-y}{x^2} \Rightarrow \frac{\partial\phi}{\partial x} = \frac{-r\sin\theta\sin\phi\cos^2\phi}{r^2\sin^2\theta\cos^2\phi}$$

$$= -\frac{\sin\phi}{r\sin\theta}$$

$$\frac{\partial}{\partial x} = \begin{pmatrix}\sin\theta\cos\phi & \frac{\cos\theta\cos\phi}{r} & \frac{-\sin\phi}{r\sin\theta}\end{pmatrix}\begin{pmatrix}\frac{\partial}{\partial r}\\ \frac{\partial}{\partial\theta}\\ \frac{\partial}{\partial\phi}\end{pmatrix}$$

$$\frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$\underset{\sim}{L}_y \doteq \frac{\hbar}{i}\Big(r\cos\theta\Big(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\Big)$$

$$-r\sin\theta\cos\phi\Big(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\Big)\Big)$$

$$\doteq \frac{\hbar}{i}\left[(\cos^2\theta+\sin^2\theta)\cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right]$$

$$\boxed{\underset{\sim}{L}_y \doteq \frac{\hbar}{i}\left[\cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right]}$$

$$\underset{\sim}{L}_\pm = \underset{\sim}{L}_x \pm i\underset{\sim}{L}_y$$

$$= \frac{\hbar}{i}\Big[-\sin\phi\frac{\partial}{\partial\theta} - \cos\phi\cot\theta\frac{\partial}{\partial\phi} \pm i\cos\phi\frac{\partial}{\partial\theta}$$

$$\mp i\sin\phi\cot\theta\frac{\partial}{\partial\phi}\Big]$$

$$= \hbar\left[\pm(\cos\phi \pm i\sin\phi)\frac{\partial}{\partial\theta} + i(\cos\phi \pm i\sin\phi)\cot\theta\frac{\partial}{\partial\phi}\right]$$

$$\underset{\sim}{L}_\pm = \pm\hbar\left[e^{\pm i\phi}\frac{\partial}{\partial\theta} \pm ie^{\pm i\phi}\cot\theta\frac{\partial}{\partial\phi}\right]$$

$$\boxed{\underset{\sim}{L}_\pm = \pm\hbar e^{\pm i\phi}\left[\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right]}$$

The idea is to take advantage of

$\underset{\sim}{L}_+|\ell\ell\rangle = 0$ $\qquad\qquad$ $\underset{\sim}{L}_-|\ell\, -\ell\rangle = 0$

$$|\ell\,\ell\rangle \doteq f_\ell(\theta)\, e^{i\ell\phi} \qquad |\ell\,{-\ell}\rangle \doteq g_\ell(\theta)\, e^{-i\ell\phi}$$

$$L_+|\ell\,\ell\rangle \doteq$$

$$\hbar e^{i\phi}\left[\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right] f_\ell(\theta) e^{i\ell\phi} \qquad \hbar e^{-i\phi}\left[\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\phi}\right] g_\ell(\theta) e^{-i\ell\phi}$$

$$= 0 \qquad\qquad = 0$$

$$\left(\frac{\partial}{\partial\theta} - \ell\cot\theta\right) f_\ell(\theta) = 0 \qquad \left(\frac{\partial}{\partial\theta} - \ell\cot\theta\right) g_\ell(\theta) = 0$$

same equation

$$f_\ell(\theta) = g_\ell(\theta)$$

so $\dfrac{df_\ell(\theta)}{d\theta} = (\ell\cot\theta)\, f_\ell(\theta)$

$$\frac{df_\ell}{f_\ell} = \ell\,\frac{\cos\theta\, d\theta}{\sin\theta} = \ell\,\frac{d(\sin\theta)}{\sin\theta}$$

$$\ln f_\ell = \ell \ln\sin\theta = \ln\sin^\ell\theta + \#$$

$$f_\ell = (\text{constant})\cdot\sin^\ell(\theta)$$

$$|\ell\,\ell\rangle \doteq N_\ell \sin^\ell\theta\, e^{i\ell\phi} \qquad |\ell\,{-\ell}\rangle \doteq N_\ell \sin^\ell\theta\, e^{-i\ell\phi}$$

can do the normalization:

$$\langle\ell\ell|\ell\ell\rangle = |N_\ell|^2 \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\, \sin^{2\ell}\theta = 1$$

(Homework ... show ...

$$N_\ell = \frac{1}{\sqrt{4\pi}} \sqrt{\frac{(2\ell+1)!!}{(2\ell)!!}} \times (-1)^\ell \leftarrow \text{convention}$$

$$n!! = n\cdot(n-2)(n-4) \ldots (2 \text{ or } 1) \;)$$

so, $$|\ell\ \ell\rangle \doteq (-1)^\ell \frac{1}{\sqrt{4\pi}} \sqrt{\frac{(2\ell+1)!!}{(2\ell)!!}} \sin^\ell\theta\, e^{+i\ell\phi}$$

$$\underset{\sim}{L}_- |\ell\ {+\ell}\rangle = \hbar\sqrt{\ell(\ell+1) - (\ell)(\ell-1)}\, |\ell\ (\ell-1)\rangle$$

$$= \hbar\sqrt{\ell(\ell+1-\ell+1)}\, |\ell\ (\ell-1)\rangle$$

$$= \hbar\sqrt{2\ell}\, |\ell\ (\ell-1)\rangle$$

$$\doteq -\hbar e^{-i\phi}\left[\frac{\partial}{\partial\theta} - i\cot\cancel{\theta\frac{\partial}{\partial\phi}}\right](-1)^\ell \frac{1}{\sqrt{4\pi}} \sqrt{\frac{(2\ell+1)!!}{(2\ell)!!}} \sin^\ell\theta\, e^{+i\ell\phi}$$

$$|\ell\ (\ell-1)\rangle \doteq (-1)^{\ell+1} \frac{1}{\sqrt{2\ell}} \frac{1}{\sqrt{4\pi}} \sqrt{\frac{(2\ell+1)!!}{(2\ell)!!}} \left[\ell\sin^{\ell-1}\theta\cos\cancel{\theta} + \ell\sin^{\ell-1}\theta\cos\theta\right] \times e^{+i(\ell-1)\phi}$$

$$|\ell\ (\ell-1)\rangle \doteq (-1)^{\ell+1} \sqrt{\frac{2\ell}{4\pi}} \sqrt{\frac{(2\ell+1)!!}{(2\ell)!!}} \sin^{\ell-1}\theta\cos\theta \times e^{i(\ell-1)\phi}$$

1 lower power $\sin\theta$ 1 higher power $\cos\theta$

so:

$$\ell=1:\quad |1\ 1\rangle \doteq (-1)\frac{1}{\sqrt{4\pi}}\sqrt{\frac{3}{2}} \sin\theta\, e^{i\phi} = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$|11\rangle \doteq Y_1^1(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$|10\rangle \doteq Y_1^0(\theta,\phi) = (-1)^2 \sqrt{\frac{2}{4\pi}}\sqrt{\frac{3}{2}}\cos\theta = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$|1\,{-1}\rangle \doteq Y_1^{-1}(\theta,\phi) = -\frac{1}{\sqrt{2}}\sqrt{\frac{3}{4\pi}}(-\sin\theta)e^{-i\phi} = \sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\phi}$$

$$|22\rangle \doteq Y_2^2(\theta,\phi) = (-1)^2 \frac{1}{\sqrt{4\pi}}\sqrt{\frac{5\cdot 3}{4\cdot 2}}\sin^2\theta\, e^{2i\phi}$$

$$= \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_2^1(\theta,\phi) = -\sqrt{\frac{\cancel{2}\cdot 2\cdot 15}{\underset{8}{\cancel{32}}\,\pi}}\;\underbrace{\sin\theta\cos\theta}\, e^{i\phi} \quad e^{im\phi}$$

alternating sign

ℓ = # powers sine + cosine

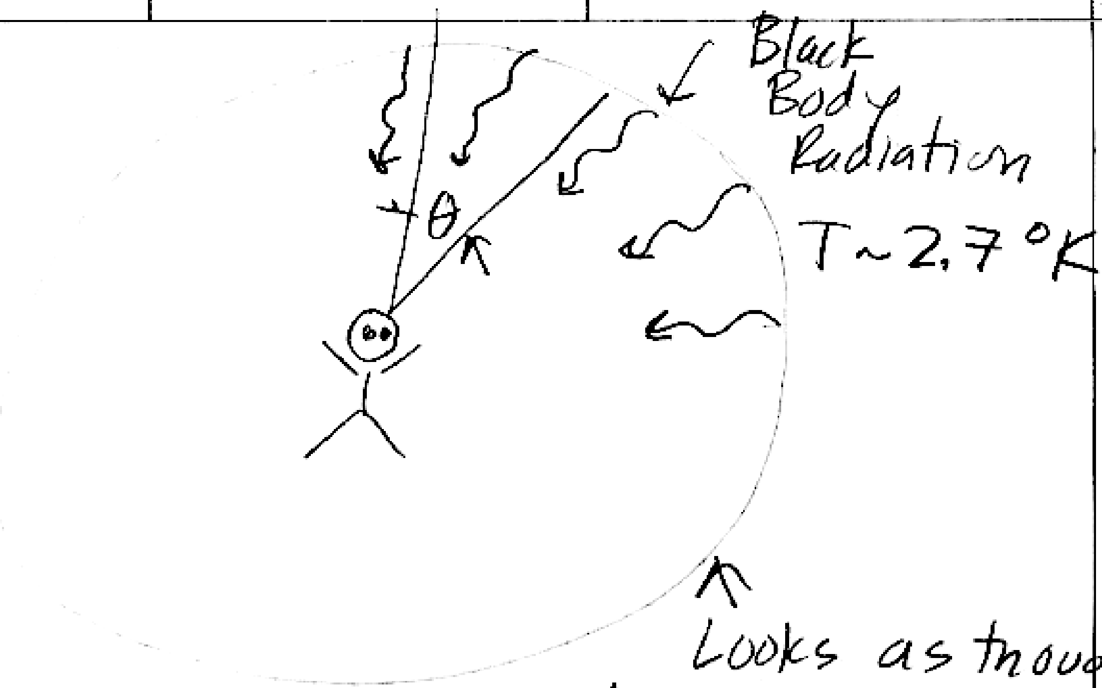
$$Y_2^0(\theta,\phi) = +\sqrt{\frac{5}{16\pi}}\,\underbrace{(3\cos^2\theta - 1)}$$

$m=0$ always has cosines alone!

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}\,; \quad Y_2^{-1}(\theta,\phi) = +\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{-i\phi}$$

$$Y_2^{-2}(\theta,\phi) = +\sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{-2i\phi}$$

The $Y_\ell^m(\theta,\phi)$ are the way one normally expands functions that are functions of θ and ϕ... like the temperature deviations of the cosmic background radiation left over from the big bang...



Looks as though it comes from a sphere ∞ far away

very tiny deviations, as a function of θ, ϕ

biggest : $(O(10^{-2} K))$ is $\propto Y_1^m(\theta,\phi)$

"earth/sun motion w/r to radiation" $\rightarrow$ Doppler Shift

then: $O(10^{-4} K) \rightarrow$ information on Big Bang:

$\Omega \sim 1$

only $\sim$ few % of universe is our kind of matter!