$$[\underset{\sim}{L}^2, \underset{\sim}{L}_j] = i\hbar \underbrace{\sum_{i \neq j}}_{\substack{\text{when } i=j, \\ \varepsilon_{ijk}=0}} \varepsilon_{ijk} (\underset{\sim}{L}_i \underset{\sim}{L}_k + \underset{\sim}{L}_k \underset{\sim}{L}_i)$$

How will this shake out? consider $j=1$

$$[\underset{\sim}{L}^2, \underset{\sim}{L}_x] = i\hbar \big(\varepsilon_{213} (\underset{\sim}{L}_2 \underset{\sim}{L}_3 + \underset{\sim}{L}_3 \underset{\sim}{L}_2)$$
$$\varepsilon_{312} (\underset{\sim}{L}_3 \underset{\sim}{L}_2 + \underset{\sim}{L}_2 \underset{\sim}{L}_3) \big)$$

$$\underset{\substack{\| \\ \varepsilon_{321}}}{\varepsilon_{213}} = -1 \qquad \underset{\substack{\| \\ \varepsilon_{123}}}{\varepsilon_{312}} = +1$$

$$[\underset{\sim}{L}^2, \underset{\sim}{L}_x] = 0 \qquad (\text{similar for } \underset{\sim}{L}_y, \underset{\sim}{L}_z)$$

<u>Do something with $\underset{\sim}{L}_x, \underset{\sim}{L}_y$ ...</u>

since $[\underset{\sim}{L}_z, \underset{\sim}{L}_x] \neq 0$, $[\underset{\sim}{L}_z, \underset{\sim}{L}_y] \neq 0$,
cannot have simultaneous eigenkets, in general.

Look at... $\underset{\sim}{L}_\pm \equiv \underset{\sim}{L}_x \pm i \underset{\sim}{L}_y$

note: $[\underset{\sim}{L}^2, \underset{\sim}{L}_\pm] = [\underset{\sim}{L}^2, \underset{\sim}{L}_x] \pm i [\underset{\sim}{L}^2, \underset{\sim}{L}_y]$
$= 0$

thus: $\underset{\sim}{L}^2 \underset{\sim}{L}_\pm |\alpha\beta\rangle = \underset{\sim}{L}_\pm \underset{\sim}{L}^2 |\alpha\beta\rangle = \alpha \underset{\sim}{L}_\pm |\alpha\beta\rangle$

this means: $\underset{\sim}{L}_\pm|\alpha\beta\rangle$ is also an eigenket of $\underset{\sim}{L}^2$, with the same eigenvalue, α.

now consider whether $\underset{\sim}{L}_\pm|\alpha\beta\rangle$ is also an eigenket of $\underset{\sim}{L}_z$. To address this, we need to evaluate the commutator of $\underset{\sim}{L}_z$ with $\underset{\sim}{L}_\pm$:

$$[\underset{\sim}{L}_z, \underset{\sim}{L}_\pm] = [\underset{\sim}{L}_z, \underset{\sim}{L}_x \pm i\underset{\sim}{L}_y]$$
$$= [\underset{\sim}{L}_z, \underset{\sim}{L}_x] \pm i[\underset{\sim}{L}_z, \underset{\sim}{L}_y]$$
$$= i\hbar\underset{\sim}{L}_y \pm i(-i\hbar\underset{\sim}{L}_x)$$
$$= i\hbar\underset{\sim}{L}_y \pm \hbar\underset{\sim}{L}_x = \pm\hbar(\underset{\sim}{L}_x \pm i\underset{\sim}{L}_y)$$

$$\boxed{[\underset{\sim}{L}_z, \underset{\sim}{L}_\pm] = \pm\hbar\underset{\sim}{L}_\pm}$$

so, $\underset{\sim}{L}_z\underset{\sim}{L}_\pm|\alpha\beta\rangle = (\underset{\sim}{L}_\pm\underset{\sim}{L}_z \pm \hbar\underset{\sim}{L}_\pm)|\alpha\beta\rangle$

$$= (\underset{\sim}{L}_\pm\beta \pm \hbar\underset{\sim}{L}_\pm)|\alpha\beta\rangle$$

$$\boxed{\underset{\sim}{L}_z\underset{\sim}{L}_\pm|\alpha\beta\rangle = (\beta\pm\hbar)\underset{\sim}{L}_\pm|\alpha\beta\rangle}$$

conclude: $\underset{\sim}{L}_\pm|\alpha\beta\rangle$ is an eigenket of $\underset{\sim}{L}_z$, but with an eigenvalue that has been raised to $\beta+\hbar$ or lowered to $\beta-\hbar$.

<u>α & β linked</u>

$$\langle\alpha\beta|\underset{\sim}{L}^2 - \underset{\sim}{L}_z^2|\alpha\beta\rangle = \alpha - \beta^2$$
$$= \langle\alpha\beta|(\underset{\sim}{L}_x^2 + \underset{\sim}{L}_y^2)|\alpha\beta\rangle \geq 0$$

so $\quad \alpha - \beta^2 \geq 0$

$$\boxed{\alpha \geq \beta^2}$$

But, can raise β with the <u>raising operator</u>:

$$\underset{\sim}{L}_+|\alpha\beta\rangle = C_+(\alpha,\beta)|\alpha\,\beta+\hbar\rangle$$

At first blush, would think that could do this repeatedly, making a ket with arbitrarily high eigenvalue of $\underset{\sim}{L}_z$.

But α still must exceed the square of that eigenvalue... a way out of this contradiction is to suggest:

$$\underset{\sim}{L}_+|\alpha\,\beta_{max}\rangle = 0$$
$$\underset{\sim}{L}_-\underset{\sim}{L}_+|\alpha\,\beta_{max}\rangle = 0$$

$$\underset{\sim}{L}_-\underset{\sim}{L}_+ = (\underset{\sim}{L}_x - i\underset{\sim}{L}_y)(\underset{\sim}{L}_x + i\underset{\sim}{L}_y)$$
$$= \underset{\sim}{L}_x^2 + \underset{\sim}{L}_y^2 + i\underbrace{(\underset{\sim}{L}_x\underset{\sim}{L}_y - \underset{\sim}{L}_y\underset{\sim}{L}_x)}_{i\hbar\underset{\sim}{L}_z}$$

$$\underset{\sim}{L}_-\underset{\sim}{L}_+ = \underset{\sim}{L}_x^2 + \underset{\sim}{L}_y^2 - \hbar\underset{\sim}{L}_z \;;\; \underset{\sim}{L}_+\underset{\sim}{L}_- = \underset{\sim}{L}_x^2 + \underset{\sim}{L}_y^2 + \hbar\underset{\sim}{L}_z$$

but $L^2 - L_z^2 = L_x^2 + L_y^2$

$L_+ L_- = L^2 - L_z^2 - \hbar L_z$; $L_- L_+ = L^2 - L_z^2 + \hbar L_z$

and $L_+ L_- |\alpha \beta_{max}\rangle = (L^2 - L_z^2 - \hbar L_z)|\alpha \beta_{max}\rangle$

$$= \alpha - \beta_{max}^2 - \hbar \beta_{max} = 0$$

$$\boxed{\alpha = \beta_{max}(\beta_{max} + \hbar)}$$

Can also examine <u>lowering</u>; can imagine starting from the state with maximum β (β_{max})

$|\alpha \beta_{max}\rangle$ $\qquad (L_+ |\alpha \beta_{max}\rangle = 0)$

$\to L_- |\alpha \beta_{max}\rangle = C_-(\alpha, \beta_{max})|\alpha, \beta_{max} - \hbar\rangle$

$\to L_-^2 |\alpha \beta_{max}\rangle = C_-(\alpha, \beta_{max} - 1)\, C_-(\alpha, \beta_{max})|\alpha, \beta_{max} - 2\hbar\rangle$

$\to$ .... eventually, β gets very <u>negative</u>, but still,

$\alpha \geq \beta_{min}^2$ $\Leftarrow$ most negative β (squaring takes out the $-$ sign)

K applications of L_-

$\to$ result in $\propto |\alpha, \beta_{min}\rangle$

$L_- |\alpha, \beta_{min}\rangle = 0$

$L_+ L_- |\alpha, \beta_{min}\rangle = (L^2 - L_z^2 + \hbar L_z)|\alpha, \beta_{min}\rangle = 0$

$$(\alpha - \beta_{min}^2 + \hbar\beta_{min}) = 0$$

$$\alpha = \beta_{min}(\beta_{min} - \hbar) = \beta_{max}(\beta_{max} + \hbar)$$

$$\beta_{min} = -\beta_{max}$$

and $\beta_{max} - \beta_{min} = 2\beta_{max} = \hbar k$

$$\boxed{\begin{aligned} &\beta_{max} = \left(\tfrac{1}{2}k\right)\cdot\hbar \quad \text{(half-integer or integer)} \\ &\text{and } \alpha = \beta_{max}(\beta_{max} + \hbar) \\ &\qquad = \hbar^2\left(\tfrac{k}{2}\right)\left(\tfrac{k}{2}+1\right) \\ &-\tfrac{1}{2}k\hbar \leq \beta \leq \tfrac{1}{2}k\hbar \end{aligned}}$$

"Full Integer": $k = 0, 2, 4, \ldots$

$$\underset{\sim}{L}_z \longrightarrow \beta = 0\hbar \quad \beta = (0, \pm 1)\hbar \quad \beta = (0, \pm 1, \pm 2)\hbar$$

$$\underset{\sim}{L}^2 \longrightarrow \alpha = 0\hbar^2 \quad 1(1+1)\hbar^2 \quad 2(2+1)\hbar^2$$

These describe solutions to the differential equations:

$$\underset{\sim}{L}_z|\alpha\beta\rangle = \beta|\alpha\beta\rangle \qquad \underset{\sim}{L}^2|\alpha\beta\rangle = \alpha|\alpha\beta\rangle$$

$$\beta = m\hbar$$

$$\frac{\hbar}{i}\frac{\partial}{\partial\phi}\Phi_m(\phi) = m\hbar\Phi_m(\phi)$$

$$-\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)Y_\ell^\ell(\theta,\phi) = \ell(\ell+1)\hbar^2 Y_\ell^\ell(\theta,\phi)$$

$$-\ell \leq m \leq \ell \qquad \text{and} \quad \alpha = \ell(\ell+1)\hbar^2$$

For these, $\Psi(\phi + 2\pi) = \Psi(\phi)$

"Half Integer" $k = 1, 3, 5, \ldots$

→ no wave function (understood yet)

now $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ (s replaces ℓ)

$$-s \le m \le s$$

To continue: $\vec{L}$ → always "orbital" with spatial wave function.

$\vec{S}$ → "intrinsic spin" no spatial wave function

"Generalization" $\vec{J}$ = <u>either</u> $\vec{L}$ <u>or</u> $\vec{S}$

$$\vec{J}^2 |jm\rangle = j(j+1)\hbar^2 |jm\rangle$$

$$J_z |jm\rangle = m\hbar |jm\rangle$$

$$[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k$$

$$J_\pm = J_x \pm i J_y$$

<u>now</u>: $J_+ |jm\rangle = \underbrace{C_+(j,m)}_{\text{want to find.}} |j\, m+1\rangle$

key point: $J_+^\dagger = (J_x + iJ_y)^\dagger = J_x^\dagger - iJ_y^\dagger$

adjoint this $= J_x - iJ_y = J_-$

$$\langle jm| J_- = C_+^*(j,m) \langle j\, m+1|$$

$$\langle jm| J_- J_+ |jm\rangle = |C_+(j,m)|^2 \langle j\, m+1|j\, m+1\rangle$$

Earlier we had deduced:

$$\underset{\sim}{J}_- \underset{\sim}{J}_+ = \underset{\sim}{J}^2 - \underset{\sim}{J}_z^2 - \hbar \underset{\sim}{J}_z$$

$$\langle jm | \underset{\sim}{J}_- \underset{\sim}{J}_+ | jm \rangle = \langle jm | (\underset{\sim}{J}^2 - \underset{\sim}{J}_z^2 - \hbar \underset{\sim}{J}_z) | jm \rangle$$

$$|C_+(j,m)|^2 = \left(j(j+1)\hbar^2 - m^2\hbar^2 - m\hbar^2\right) \overset{1}{\cancel{\langle jm | jm \rangle}}$$

$$= \left[j(j+1) - m(m+1) \right] \hbar^2$$

choosing the phase as 1

$$\boxed{C_+(j,m) = \sqrt{j(j+1) - m(m+1)}\,\hbar = \sqrt{(j-m)(j+m+1)}\,\hbar}$$

<u>Similarly</u>:

$$\underset{\sim}{J}_- | jm \rangle = C_-(j,m) | j\, m-1 \rangle$$

$$\langle jm | \underset{\sim}{J}_+ \underset{\sim}{J}_- | jm \rangle = |C_-(j,m)|^2 \overset{1}{\cancel{\langle jm-1 | jm-1 \rangle}}$$

$$\hookrightarrow \underset{\sim}{J}_+ \underset{\sim}{J}_- = \underset{\sim}{J}^2 - \underset{\sim}{J}_z^2 + \hbar \underset{\sim}{J}_z$$

so $$|C_-(j,m)|^2 = \left[j(j+1) - m(m-1) \right] \hbar^2$$

$$\boxed{C_-(j,m) = \sqrt{j(j+1) - m(m-1)}\,\hbar = \sqrt{(j+m)(j-m+1)}\,\hbar}$$

another way to say this:

$$\langle j'm' | \underset{\sim}{J}_+ | j\, m \rangle = 0 \quad \text{unless } j = j'$$

$$= \delta_{j'j} \sqrt{j(j+1) - m(m+1)}\, \delta_{m'\, m+1}\, \hbar$$

$$\langle j'm' | \underset{\sim}{J}_- | jm \rangle = \delta_{j'j} \sqrt{j(j+1) - m(m-1)}\, \delta_{m'\, m-1}\, \hbar$$