so $$\boxed{L_z = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = \frac{\hbar}{i}\frac{\partial}{\partial \phi}}$$

Rotational Invariance of H

$$\underbrace{H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + V(\sqrt{x^2+y^2+z^2})}$$

pretty obviously depends on only the <u>length</u> of vectors; what this means quantum mechanically is

$$\underbrace{U^\dagger(R)}_{\text{adjoint of rotation operator (arbitrary rotation)}} \; H \; \underbrace{U(R)}_{\text{rotation operator}} = H$$

(not a proof, an argument)

take rotation about z axis, $\phi \to 0, = \varepsilon_z$

$$U(\varepsilon_z \hat{z}) \cong \mathbb{1} - \frac{i\varepsilon_z}{\hbar} L_z$$

$$\left(\mathbb{1} + \frac{i\varepsilon_z}{\hbar}L_z\right) H \left(\mathbb{1} - \frac{i\varepsilon_z}{\hbar}L_z\right) = H$$

neglecting $O(\varepsilon^2)$

$$\frac{i\varepsilon_z}{\hbar}[L_z, H] = 0$$

$$[L_z, H] = 0$$

Meaning Rotational Invariance (about z) implies $[L_z, H] = 0$ which implies that

i) simultaneous eigenkets of L_z and H

ii) $\frac{d\langle L_z\rangle}{dt} = 0$

true for all l_z. For example, l_z must be real-valued since Hermitian. More generally,

$$\langle\psi_2|\underset{\sim}{L}_z|\psi_1\rangle = \langle\psi_1|(\underset{\sim}{L}_z^+ = \underset{\sim}{L}_z)|\psi_2\rangle^*$$

or $$\int_0^\infty \cancel{\rho\, d\rho\, R_2^*(\rho) R_1(\rho)} \int_0^{2\pi} d\phi\, e^{\frac{-i l_{z2}\phi}{\hbar}} \frac{\hbar}{i}\frac{\partial}{\partial\phi} e^{\frac{i l_{z1}\phi}{\hbar}}$$

$$= \int_0^\infty \cancel{\rho\, d\rho\, R_1(\rho) R_2^*(\rho)} \int e^{\frac{+i l_{z1}\phi}{\hbar}} \frac{\hbar}{-i}\frac{\partial}{\partial\phi} e^{\frac{-i l_{z2}\phi}{\hbar}} d\phi$$

or $$\underbrace{\int_0^{2\pi} d\phi\, e^{\frac{-i l_{z2}\phi}{\hbar}} \frac{\partial}{\partial\phi} e^{\frac{i l_{z1}\phi}{\hbar}}}_{\text{integrate by parts:}} = -\int_0^{2\pi} d\phi\, e^{\frac{i l_{z1}\phi}{\hbar}} \frac{\partial}{\partial\phi} e^{\frac{-i l_{z2}\phi}{\hbar}}$$

$$u = e^{\frac{-i l_{z2}\phi}{\hbar}} \qquad dv = d\phi \frac{\partial}{\partial\phi} e^{\frac{i l_{z1}\phi}{\hbar}}$$

$$du = \frac{\partial}{\partial\phi} e^{\frac{-i l_{z2}\phi}{\hbar}} d\phi \qquad V = e^{\frac{i l_{z1}\phi}{\hbar}}$$

$$\text{R.H.S.} = e^{\frac{i}{\hbar}(l_{z1}-l_{z2})\phi}\Big|_0^{2\pi} - \int_0^{2\pi} d\phi\, e^{\frac{i l_{z1}\phi}{\hbar}} \frac{\partial}{\partial\phi} e^{\frac{-i l_{z2}\phi}{\hbar}}$$

$$\text{LHS} = \qquad 0 \qquad - \qquad ||$$

need: $$e^{\frac{i}{\hbar}(l_{z1}-l_{z2})\phi}\Big|_0^{2\pi} = 0$$

$$e^{\frac{i}{\hbar}(l_{z1}-l_{z2})2\pi} = 1$$

so $$l_{z1} - l_{z2} = \text{integer} \cdot \hbar$$

note: since $[\underset{\sim}{L}_x, \underset{\sim}{L}_z] \neq 0$; $[\underset{\sim}{L}_y, \underset{\sim}{L}_z] \neq 0$
cannot guarantee simultaneous eigenkets of all three ...

Eigenkets of $\underset{\sim}{L}_z$

want $\underset{\sim}{L}_z |l_z\rangle = l_z |l_z\rangle$

Project into coordinate basis: here we imagine that we are in 2-d space

$$\rho = \sqrt{x^2+y^2} \Rightarrow \underset{\sim}{\rho} \equiv \sqrt{\underset{\sim}{X}^2+\underset{\sim}{Y}^2}$$
$$\phi = \tan^{-1}\left(\frac{y}{x}\right) \Rightarrow \underset{\sim}{\phi} \equiv \tan^{-1}\left(\frac{\underset{\sim}{Y}}{\underset{\sim}{X}}\right)$$

$$\underset{\sim}{\rho}|\rho,\phi\rangle = \rho|\rho,\phi\rangle \qquad \underset{\sim}{\phi}|\rho,\phi\rangle = \phi|\rho,\phi\rangle$$

then $$\underbrace{\langle\rho,\phi|\underset{\sim}{L}_z|l_z\rangle}_{\frac{\hbar}{i}\frac{\partial}{\partial\phi}\psi_{l_z}(\rho,\phi)} = l_z \underbrace{\langle\rho,\phi|l_z\rangle}_{\psi_{l_z}(\rho,\phi)}$$

or $$\frac{\hbar}{i}\frac{\partial\psi_{l_z}(\rho,\phi)}{\psi_{l_z}(\rho,\phi)} = l_z\,\partial\phi$$

$$\ln \psi_{l_z} = \frac{i l_z}{\hbar}\phi + \text{constant}$$

could be a function of ρ, $\ln R(\rho)$

$$\boxed{\psi_{l_z} = R(\rho)\, e^{\frac{i l_z}{\hbar}\phi}}$$

Condition on $\frac{i l_z}{\hbar}$ results from requirement that $\underset{\sim}{L}_z$ be Hermitian, that is <u>not</u>

<u>IF</u> we also require that:

$$\psi_{\ell z}(\rho,\phi) = \psi_{\ell z}(\rho,\phi+2\pi) \quad \text{(single valued wave function)}$$

Then $\boxed{\ell_z = (\text{integer})\,\hbar}$

and, Hermitian requirement is also satisfied.

<u>Deeper discussion</u>: why <u>not</u>
$\ell_z = (\text{integer} + 0.31275)\hbar$?

Real reason is this leads to a singularity in probability flux, I recall.

Finally, $\ell_z = m\hbar \qquad m = 0, \pm 1, \pm 2, \ldots$

Convenient to define $\qquad m \rightarrow$ "magnetic quantum #"

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}$$

then $\langle m' | m \rangle = \int_0^{2\pi} \Phi^*_{m'}(\phi)\,\Phi_m(\phi)\,d\phi$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\phi\, e^{i(m-m')\phi}$$

when $m \neq m'$ $\qquad = \frac{1}{2\pi}\frac{1}{i(m-m')}\left[e^{2\pi i(m-m')} - 1\right] = 0$

when $m = m'$ $\qquad = \frac{1}{2\pi}\cdot 2\pi = 1$

so $\int_0^{2\pi} \Phi^*_{m'}(\phi)\,\Phi_m(\phi)\,d\phi = \delta_{m'm}$

2-d Hamiltonian

$$\underset{\sim}{H} = \frac{1}{2\mu}\left(p_x^2 + p_y^2\right) + V\left(\sqrt{\underset{\sim}{x}^2+\underset{\sim}{y}^2}\right)$$ ← assume independent of ϕ

use μ, since m taken by "magnetic quantum #"

Classically:

$$p_x^2 + p_y^2 = \frac{(xp_x + yp_y)^2}{x^2+y^2} + \frac{(xp_y - yp_x)^2}{x^2+y^2}$$

check
$$= \frac{x^2p_x^2 + 2xyp_xp_y + y^2p_y^2 + x^2p_y^2 - 2xyp_xp_y + y^2p_x^2}{x^2+y^2}$$

$$= \frac{(x^2+y^2)p_x^2 + (x^2+y^2)p_y^2}{x^2+y^2} = p_x^2 + p_y^2$$

but moving to polar coordinates:

1) $\rho^2 = x^2+y^2$

2) ρ-component of $\vec{p} \Rightarrow \frac{xp_x + yp_y}{\rho} = p_\rho$

3) $L_z = xp_y - yp_x$

so
$$p_x^2 + p_y^2 = p_\rho^2 + \frac{L_z^2}{\rho^2}$$

$$\rightarrow \underset{\sim}{H} = \frac{1}{2\mu}\left(p_\rho^2 + \frac{L_z^2}{\rho^2}\right) + V\left(\sqrt{\underset{\sim}{x}^2+\underset{\sim}{y}^2}\right)$$

Representation of P_ρ^2 is challenging.... (can skip explicit stuff).

$$\vec{R} \doteq \hat{i}\frac{\hbar}{i}\frac{\partial}{\partial x} + \hat{j}\frac{\hbar}{i}\frac{\partial}{\partial y}$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial \phi}{\partial x} & \frac{\partial \rho}{\partial x} \\ \frac{\partial \phi}{\partial y} & \frac{\partial \rho}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial \rho} \end{pmatrix}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) \qquad \frac{\partial \phi}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$= -\frac{\rho\sin\phi}{\rho^2} = -\frac{\sin\phi}{\rho}$$

$$\frac{\partial \phi}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x} = \frac{x}{x^2+y^2} = \frac{\cos\phi}{\rho}$$

$$\rho = \sqrt{x^2+y^2} \qquad \frac{\partial \rho}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \cos\phi$$

$$\frac{\partial \rho}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \sin\phi$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \begin{pmatrix} -\frac{\sin\phi}{\rho} & \cos\phi \\ \frac{\cos\phi}{\rho} & \sin\phi \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial \rho} \end{pmatrix}$$

$$\frac{\partial}{\partial x} = -\frac{\sin\phi}{\rho}\frac{\partial}{\partial \phi} + \cos\phi\frac{\partial}{\partial \rho}$$

$$\frac{\partial}{\partial y} = \frac{\cos\phi}{\rho}\frac{\partial}{\partial \phi} + \sin\phi\frac{\partial}{\partial \rho}$$

$$\vec{R} \doteq \frac{\hbar}{i}\left(\hat{i}\left(-\frac{\sin\phi}{\rho}\frac{\partial}{\partial\phi} + \cos\phi\frac{\partial}{\partial\rho}\right) + \hat{j}\left(\frac{\cos\phi}{\rho}\frac{\partial}{\partial\phi} + \sin\phi\frac{\partial}{\partial\rho}\right)\right)$$

$$p^2 \doteq -\hbar^2 \Big(\Big(-\frac{\sin\phi}{\rho}\frac{\partial}{\partial\phi} + \cos\phi \frac{\partial}{\partial\rho}\Big)\Big(-\frac{\sin\phi}{\rho}\frac{\partial}{\partial\phi} + \cos\phi\frac{\partial}{\partial\rho}\Big)$$
$$+ \Big(\frac{\cos\phi}{\rho}\frac{\partial}{\partial\phi} + \sin\phi\frac{\partial}{\partial\rho}\Big)\Big(\frac{\cos\phi}{\rho}\frac{\partial}{\partial\phi} + \sin\phi\frac{d}{d\rho}\Big)\Big)$$

$$\doteq -\hbar^2\Big[+\frac{\sin\phi\cos\phi}{\rho^2}\frac{\partial}{\partial\phi} + \frac{\sin^2\phi}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\sin^2\phi}{\rho}\frac{\partial}{\partial\rho} - \frac{\sin\phi\cos\phi}{\rho}\frac{\partial^2}{\partial\phi\partial\rho}$$
$$+ \frac{\sin\phi\cos\phi}{\rho^2}\frac{\partial}{\partial\phi} - \frac{\sin\phi\cos\phi}{\rho}\frac{\partial^2}{\partial\rho\, d\phi} + \cos^2\phi\frac{\partial^2}{\partial\rho^2}$$
$$- \frac{\sin\phi\cos\phi}{\rho^2}\frac{\partial}{\partial\phi} + \frac{\cos^2\phi}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\cos^2\phi}{\rho}\frac{d}{d\rho} + \frac{\sin\phi\cos\phi}{\rho}\frac{\partial^2}{\partial\phi\partial\rho}$$
$$- \frac{\sin\phi\cos\phi}{\rho^2}\frac{\partial}{\partial\phi} + \frac{\sin\phi\cos\phi}{\rho}\frac{\partial^2}{\partial\rho\, d\phi} + \sin^2\phi\frac{\partial}{\partial\rho^2}$$

(circled terms don't cancel)

$$\doteq -\hbar^2\Big[\underbrace{\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\Big] \qquad (12.3.11$$

p. 315)

This (after inclusion of $-\hbar^2$) is representation of p_ρ^2!

this (after inclusion of $-\hbar^2$) is representation of L_z^2!

$$\underset{\sim}{H} = \frac{1}{2\mu}\Big(p_\rho^2 + \frac{L_z^2}{\rho^2}\Big) + V\big(\sqrt{\underset{\sim}{X}^2 + \underset{\sim}{Y}^2}\big)$$

$$\doteq \qquad \doteq \qquad \doteq$$

$$= \frac{-\hbar^2}{2\mu}\Big(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\Big) + V(\rho)$$

So, the eigenvalue equation for energy is:

$$\underset{\sim}{H}|E\rangle = E|E\rangle$$

$$\parallel$$

$$-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+V(\rho)\right)\psi_E(\rho,\phi)$$

$$= E\psi_E(\rho,\phi)$$

Try: $\psi_E(\rho,\phi) = R(\rho)\,\Phi_m(\phi)$

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi}$$

$\Rightarrow$ capitialize on the fact that $[\underset{\sim}{H},\underset{\sim}{L_z}]=0$ (since $V(\rho,\phi)$ independent of ϕ, meaning $V(\rho,\phi)$ is rotationally invariant).

$\Rightarrow$ simultanews eigenkets of $\underset{\sim}{H}$, $\underset{\sim}{L_z}$

$$-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+V(\rho)\right)R(\rho)\,\Phi_m(\phi)$$

$$= \cancel{\Phi_m(\phi)}\left[-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{m^2}{\rho^2}\right)+V(\rho)\right]R(\rho)$$

$$= E\,\cancel{\Phi_m(\phi)}\,R(\rho)$$

"Radial Equation" gives $E(m)$: (p. 316, 12.3.12)

$$\boxed{\left[-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{m^2}{\rho^2}\right)+V(\rho)\right]R_{Em}(\rho) = E(m)\,R_{Em}(\rho)}$$