Translation Operator

(p. 279, chap. 11)

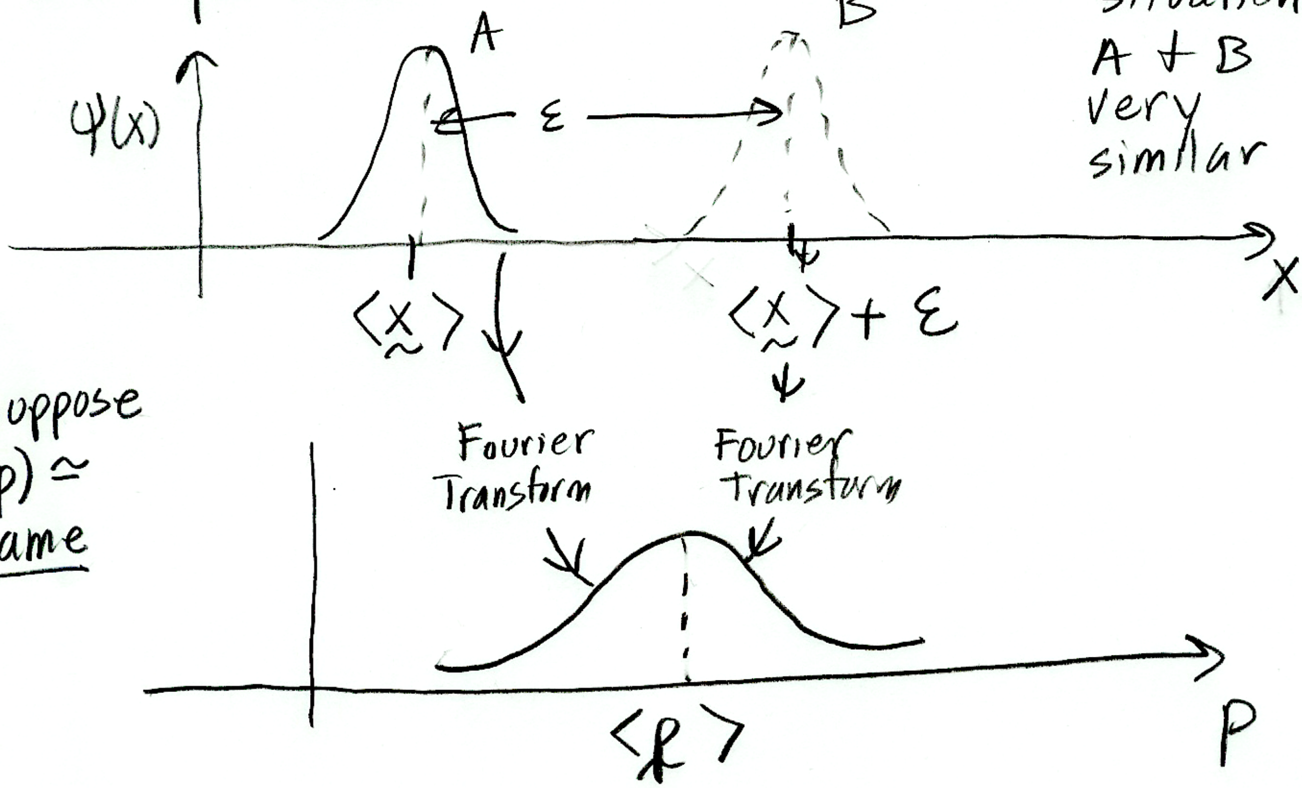
motivation: in "free space" (say, 1-d)

$$\underset{\sim}{H} = \frac{\underset{\sim}{p}^2}{2m} + \cancel{V(\underset{\sim}{x})}^{\,0}$$

a translation doesn't make much difference.

In particular:

situation A + B very similar



suppose $\varphi(p) \simeq$ same

Observations:
- physically, OK
- mathematically, the $\varphi(p)$'s

that result from the two fourier transforms will differ by a factor like $e^{i\phi}$.

- the wavepacket in p-space is obviously not an eigenstate of $\underset{\sim}{p}$; $\therefore$ not an eigenket of $\underset{\sim}{p}^2$, so not an eigenstate of $\underset{\sim}{H}$. So, the wavepacket in x, $\psi(x)$, spreads out in time.

- Back to defining translation ... want the concept to work on non-eigenstates of $\underset{\sim}{H}$ ... don't use invariance of energy as a function of position (yet).

Active View

$$\text{situation B}: \; |\psi_\varepsilon\rangle = \underset{\sim}{T}(\varepsilon)|\psi\rangle \leftarrow \text{situation A}$$

definition of translation operator

want:

$$\langle\psi_\varepsilon|\underset{\sim}{x}|\psi_\varepsilon\rangle = \langle\psi|\underset{\sim}{x}|\psi\rangle + \varepsilon$$

$\langle\psi|\underset{\sim}{T}^\dagger(\varepsilon)$ $\qquad$ $\underset{\sim}{T}(\varepsilon)|\psi\rangle$

or $$\langle\psi|\underset{\sim}{T}^\dagger(\varepsilon)\underset{\sim}{x}\,\underset{\sim}{T}(\varepsilon)|\psi\rangle = \langle\psi|\underset{\sim}{x}|\psi\rangle + \varepsilon$$

"Actively Push State": $\underbrace{\underset{\sim}{T}(\varepsilon)|\psi\rangle} = |\psi_\varepsilon\rangle$

(leave operators alone)

Passive View

group the terms differently (since only expectation values matter, OK to view this way).

$$\langle\psi|\underbrace{\underset{\sim}{T}^\dagger(\varepsilon)\underset{\sim}{x}\,\underset{\sim}{T}(\varepsilon)}|\psi\rangle$$

view this as a new $\underset{\sim}{x}$ operater

and $$\underset{\sim}{T}^\dagger(\varepsilon)\,\underset{\sim}{x}\,\underset{\sim}{T}(\varepsilon) = \underset{\sim}{x} + \varepsilon\cdot\underset{\sim}{1}$$

Passive viewpoint: • state does not get pushed (left alone)
• operator is adjusted.

Physically, these are equivalent: cannot distinguish whether the state or system is shifted forward by ε and the universe (including the apparatus that would measure $\underset{\sim}{x}$) is unchanged [THE ACTIVE VIEW]; or the state or system is unchanged and the universe (including $\underset{\sim}{x}$-measuring apparatus) is shifted backward by ε [THE PASSIVE VIEW].

Both views are useful + used! Be on guard; there are no conventions or agreements about which to use; both are everywhere

NOTE: define $\underset{\sim}{T}(\varepsilon)$ such that:

$$\langle \psi_\varepsilon | \underset{\sim}{R} | \psi_\varepsilon \rangle = \langle \psi | \underset{\sim}{R} | \psi \rangle$$

or $$\langle \psi | T^\dagger(\varepsilon) \underset{\sim}{R} \underset{\sim}{T}(\varepsilon) | \psi \rangle = \langle \psi | \underset{\sim}{R} | \psi \rangle$$

How $\underset{\sim}{T}(\varepsilon)$ changes the wavefunction

$$\underset{\sim}{T}(\varepsilon)|x\rangle = |x+\varepsilon\rangle$$

$$\underset{\sim}{x}|x\rangle = x|x\rangle \qquad \underset{\sim}{x}|x\rangle = (x+\varepsilon)|x+\varepsilon\rangle$$

$\psi(x) = \langle x | \psi \rangle$ represents $|\psi\rangle$

What wavefunction represents $\underset{\sim}{T}(\varepsilon)|\psi\rangle$?

$$\underset{\sim}{I}(\varepsilon)|\psi\rangle = \underset{\sim}{I}(\varepsilon)\int_{-\infty}^{\infty}|x\rangle\langle x|\psi\rangle dx = \int_{-\infty}^{\infty}|x+\varepsilon\rangle\langle x|\psi\rangle dx$$

insert $\underset{\sim}{1} = \int_{-\infty}^{\infty} dx\, |x\rangle\langle x|$

Change Variables!

$x' \equiv x+\varepsilon$

$x = x'-\varepsilon$

$dx = dx'$

$$\underset{\sim}{I}(\varepsilon)|\psi\rangle = \int_{-\infty}^{\infty}|x'\rangle\langle x'-\varepsilon|\psi\rangle dx' = \int_{-\infty}^{\infty}|x\rangle\underbrace{\langle x-\varepsilon|\psi\rangle}_{\psi(x-\varepsilon)}dx$$

$x = x'$
change back

or, $\underset{\sim}{I}(\varepsilon)|\psi\rangle$ is represented by $\psi(x-\varepsilon)$

For example, if $|\psi\rangle \doteq A e^{-x^2/2a^2}$

then $\underset{\sim}{I}(\varepsilon)|\psi\rangle \doteq A e^{-(x-\varepsilon)^2/2a^2}$

x must <u>overcome</u> ε

What about p?

$$\langle\psi_\varepsilon|p|\psi_\varepsilon\rangle = \int_{-\infty}^{\infty} dx\, \psi_\varepsilon^*(x)\left(\frac{\hbar}{i}\frac{d}{dx}\right)\psi_\varepsilon(x)$$

$$= \int_{-\infty}^{\infty} dx\, \psi^*(x-\varepsilon)\left(\frac{\hbar}{i}\frac{d}{dx}\right)\psi(x-\varepsilon)$$

change variables: $x' = x-\varepsilon$

$dx' = dx$

$$\frac{df}{dx} = \frac{df}{dx'}\left(\frac{dx'}{dx} = 1\right)$$

$$\frac{d}{dx} = \frac{d}{dx'}$$

and so $\langle \psi_\varepsilon | \underset{\sim}{p} | \psi_\varepsilon \rangle = \int_{-\infty}^{\infty} dx' \, \psi^*(x') \left(\frac{\hbar}{i} \frac{d}{dx'} \right) \psi(x')$

$= \langle \psi | \underset{\sim}{p} | \psi \rangle$

$\underset{\sim}{T}(\varepsilon)$ is unitary

$$\langle x' | x \rangle = \delta(x' - x)$$

note:
$$\langle x' | \underset{\sim}{T}^{\dagger}(\varepsilon) \underset{\sim}{T}(\varepsilon) | x \rangle = \langle x' + \varepsilon | x + \varepsilon \rangle$$
$$= \delta(x' + \varepsilon - x - \varepsilon) = \delta(x' - x)$$

<u>conclude</u>: $\underset{\sim}{T}^{\dagger}(\varepsilon) \underset{\sim}{T}(\varepsilon) = \underset{\sim}{\mathbb{1}}$; little work $\underset{\sim}{T}(\varepsilon)\underset{\sim}{T}^{\dagger}(\varepsilon) = \underset{\sim}{\mathbb{1}}$

The Generator of $\underset{\sim}{T}(\varepsilon)$

Let's suggest $\underset{\sim}{T}(\varepsilon) = \underset{\sim}{\mathbb{1}} - \frac{i\varepsilon}{\hbar} \underset{\sim}{G}$ as $\varepsilon \to 0$

Why?
- know $\underset{\sim}{T}(\varepsilon) \to \underset{\sim}{\mathbb{1}}$ as $\varepsilon \to 0$
- $\frac{i}{\hbar}$ just arbitrary, but nice.

now, to order ε: $\underset{\sim}{T}^{\dagger}(\varepsilon) \underset{\sim}{T}(\varepsilon) = \underset{\sim}{\mathbb{1}}$

$$\left(\underset{\sim}{\mathbb{1}} - \frac{i\varepsilon}{\hbar} \underset{\sim}{G} \right)^{\dagger} \left(\underset{\sim}{\mathbb{1}} - \frac{i\varepsilon}{\hbar} \underset{\sim}{G} \right) = \underset{\sim}{\mathbb{1}}$$

$$\cancel{\underset{\sim}{\mathbb{1}}} + \frac{i\varepsilon}{\hbar} \underset{\sim}{G}^{\dagger} - \frac{i\varepsilon}{\hbar} \underset{\sim}{G} = \cancel{\underset{\sim}{\mathbb{1}}} \qquad \boxed{\underset{\sim}{G}^{\dagger} = \underset{\sim}{G}}$$

For the -ation operator to be unitary, its generator ~~must~~ be Hermitian.

The Generator of $\underset{\sim}{T}(a)$ is the Momentum

$$\langle x|\underset{\sim}{T}(\varepsilon)|\psi\rangle = \psi(x-\varepsilon)$$

$$\langle x|(\underset{\sim}{1} - \frac{i\varepsilon}{\hbar}\underset{\sim}{G})|\psi\rangle = \psi(x) - \varepsilon \frac{d\psi}{dx}\Big|_x$$

$$\psi(x) - \frac{i\varepsilon}{\hbar}\langle x|\underset{\sim}{G}|\psi\rangle = \psi(x) - \varepsilon \frac{d\psi}{dx}\Big|_x$$

$$\langle x|\underset{\sim}{G}|\psi\rangle = \frac{\hbar}{i}\frac{d\psi}{dx}\Big|_x$$

$$\boxed{\underset{\sim}{G} = p\,!}$$

Return to the Hamiltonian

when will $\langle\psi|\underset{\sim}{H}|\psi\rangle = \langle\psi_\varepsilon|\underset{\sim}{H}|\psi_\varepsilon\rangle$?

Physical answer: when $\underset{\sim}{H}$ = independent of $\underset{\sim}{\hat{x}}$.

Quantum answer:

when $\langle\psi|\underset{\sim}{H}|\psi\rangle = \langle\psi|\underset{\sim}{T}^{\dagger}(\varepsilon)\underset{\sim}{H}\,\underset{\sim}{T}(\varepsilon)|\psi\rangle$

since $|\psi\rangle$ ~~is~~ arbitrary, means

$$\underset{\sim}{H} = \underset{\sim}{T}^{\dagger}(\varepsilon)\, \underset{\sim}{H}\, \underset{\sim}{T}(\varepsilon)$$

since $\underset{\sim}{T}(\varepsilon)$ is unitary, $\underset{\sim}{T}(\varepsilon)\underset{\sim}{T}^{\dagger}(\varepsilon) = \underset{\sim}{\mathbb{1}}$, and

so $$\underset{\sim}{T}(\varepsilon)\underset{\sim}{H} = \underbrace{\underset{\sim}{T}(\varepsilon)\underset{\sim}{T}^{\dagger}(\varepsilon)}_{\underset{\sim}{\mathbb{1}}}\underset{\sim}{H}\,\underset{\sim}{T}(\varepsilon)$$

or $$\underset{\sim}{T}(\varepsilon)\underset{\sim}{H} - \underset{\sim}{H}\,\underset{\sim}{T}(\varepsilon) = 0$$

$$\boxed{\text{when } [\underset{\sim}{T}(\varepsilon), \underset{\sim}{H}] = 0, \ \langle \underset{\sim}{H} \rangle \text{ is translation independent}}$$

take it further... as $\varepsilon \to 0$,

$$\underset{\sim}{T}(\varepsilon) \simeq \underset{\sim}{\mathbb{1}} - \frac{i\varepsilon}{\hbar} p$$

$$\left[\underset{\sim}{\mathbb{1}} - \frac{i}{\hbar}\varepsilon p, \underset{\sim}{H}\right] = -\frac{i\varepsilon}{\hbar}[p, \underset{\sim}{H}] = 0$$

examples:

(1) $\underset{\sim}{H} = \frac{p^2}{2m}$ or $= \sqrt{(mc^2)^2 + (cp)^2}$

then $[p, \underset{\sim}{H}] = 0 \Rightarrow \langle \underset{\sim}{H} \rangle$ translation ind.

(2) $\underset{\sim}{H} = \frac{p^2}{2m} + V(\underset{\sim}{x})$

$$\uparrow$$

$[p, V(\underset{\sim}{x})] \neq 0$! $\langle \underset{\sim}{H} \rangle$ NOT TRANSLATION INDEPENDENT