The normalization & raising & lowering

$$\underset{\sim}{a}|n\rangle = C_n|n-1\rangle \qquad \left(\begin{array}{c}\text{replaced } \varepsilon \text{ by} \\ n,\ \varepsilon = n+\frac{1}{2}\end{array}\right)$$

$\varepsilon = n+\frac{1}{2}$ $\qquad$ $\varepsilon = n+\frac{1}{2}-1 = n-\frac{1}{2}$

$$\underset{\sim}{a}^{+}|n\rangle = C_{n+1}|n+1\rangle$$

adjoint $\langle n|\underset{\sim}{a} = C^{*}_{n+1}\langle n+1|$

so $\langle n|\underbrace{\underset{\sim}{a}\,\underset{\sim}{a}^{+}}|n\rangle = C_{n+1}C^{*}_{n+1}\overbrace{\langle n+1|n+1\rangle}^{1}$

recall $\hat{\underset{\sim}{H}} = \underset{\sim}{a}^{+}\underset{\sim}{a}+\frac{1}{2} = \underset{\sim}{a}\,\underset{\sim}{a}^{+}-\frac{1}{2}$

$\underset{\sim}{a}\,\underset{\sim}{a}^{+} = \hat{\underset{\sim}{H}}+\frac{1}{2}$

so $\langle n|(\hat{\underset{\sim}{H}}+\frac{1}{2})|n\rangle = |C_{n+1}|^2$

$\hat{\underset{\sim}{H}}|n\rangle = (n+\frac{1}{2})|n\rangle$ since $|n\rangle$ is an eigenket

so $\langle n|(n+1)|n\rangle = |C_{n+1}|^2$

$(n+1)\underbrace{\langle n|n\rangle}_{1} = |C_{n+1}|^2$

$\sqrt{n+1}\,e^{i\phi} = C_{n+1}$ $\Leftarrow$ ϕ a real #

take $\phi = 0$

$C_{n+1} = \sqrt{n+1}$

or $\boxed{\underset{\sim}{a}^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle}$

also $\boxed{\underset{\sim}{a}|n\rangle = \sqrt{n}\,|n-1\rangle}$ in book, p.206

$\boxed{\text{Number Operator}}$

$$\underset{\sim}{\hat{H}}|n\rangle = (n+\tfrac{1}{2})|n\rangle$$

$$\underset{\sim}{\hat{H}} = \underset{\sim}{a}^{+}\underset{\sim}{a} + \tfrac{1}{2}$$

so $(\underset{\sim}{a}^{+}\underset{\sim}{a} + \tfrac{1}{2})|n\rangle = (n+\tfrac{1}{2})|n\rangle$

$$\underset{\sim}{a}^{+}\underset{\sim}{a}|n\rangle = n|n\rangle$$

so, named the <u>number</u> operator $\underset{\sim}{N}$

$$\underset{\sim}{N} \equiv \underset{\sim}{a}^{+}\underset{\sim}{a}$$

another way: $\underset{\sim}{a}^{+}\underset{\sim}{a}|n\rangle = \underset{\sim}{a}^{+}\sqrt{n}\,|n-1\rangle = \sqrt{n}\,\underset{\sim}{a}^{+}|n-1\rangle$

$$= \sqrt{n}\,\sqrt{n-1+1}\,|n\rangle$$

$$\underset{\sim}{a}^{+}\underset{\sim}{a}|n\rangle = n|n\rangle$$

Matrix Elements of $\underset{\sim}{a}$, $\underset{\sim}{a}^{+}$

$$\langle n'|\underset{\sim}{a}|n\rangle = \langle n'|\sqrt{n}\,|n-1\rangle = \sqrt{n}\,\langle n'|n-1\rangle$$

$\boxed{\langle n'|\underset{\sim}{a}|n\rangle = \sqrt{n}\,\delta_{n',n-1}}$ 7.4.26, p. 207

$$\langle n'|\underset{\sim}{a}^{+}|n\rangle = \langle n'|\sqrt{n+1}\,|n+1\rangle = \sqrt{n+1}\,\langle n'|n+1\rangle$$

$$\boxed{\langle n'|\underset{\sim}{a}^{+}|n\rangle = \sqrt{n+1}\,\delta_{n',n+1}}$$

these can be visualized as matrices.

$$\langle n' | \underset{\sim}{a} | n \rangle = \sqrt{n}\, \delta_{n', n-1}$$

$$\begin{array}{c|ccccccc}
 & n=0 & 1 & 2 & 3 & 4 & 5 & 6 \ldots \\
\hline
n'=0 & 0 & \sqrt{1} & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & \sqrt{4} & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & \sqrt{5} & 0 \\
5 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} \\
\vdots & & & & & & &
\end{array}$$

$$\langle n' | \underset{\sim}{a}^{+} | n \rangle = \sqrt{n+1}\, \delta_{n', n+1}$$

$$\begin{array}{c|ccccccc}
n'=0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \sqrt{1} & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & \sqrt{4} & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & \sqrt{5} & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} & 0 \ \ldots
\end{array}$$

These are pretty clearly not Hermitian.
What about $\underset{\sim}{x}$ and $\underset{\sim}{p}$?

$$\underset{\sim}{a} = \frac{1}{\sqrt{2}}\left[\frac{\underset{\sim}{x}}{b} + \frac{ib}{\hbar}\underset{\sim}{p}\right] \qquad b = \sqrt{\frac{\hbar}{m\omega}}$$

$$\underset{\sim}{a}^{+} = \frac{1}{\sqrt{2}}\left[\frac{\underset{\sim}{x}}{b} - \frac{ib}{\hbar}\underset{\sim}{p}\right]$$

$$\underset{\sim}{a} + \underset{\sim}{a}^{+} = \frac{2}{\sqrt{2}}\frac{\underset{\sim}{x}}{b} \Rightarrow \underset{\sim}{x} = \frac{b}{\sqrt{2}}(\underset{\sim}{a} + \underset{\sim}{a}^{+})$$

$$\underset{\sim}{a} - \underset{\sim}{a}^{+} = \frac{2}{\sqrt{2}}\frac{ib}{\hbar}\underset{\sim}{p} \Rightarrow \underset{\sim}{p} = \frac{i\hbar}{\sqrt{2}b}(\underset{\sim}{a}^{+} - \underset{\sim}{a})$$

$\langle n'|\underset{\sim}{x}|n\rangle =$

	$n=0$	1	2	3	4	5	6	
$n'=0$	0	$\sqrt{1}$	0	0	0	0	0	...
1	$\sqrt{1}$	0	$\sqrt{2}$	0	0	0	0	
2	0	$\sqrt{2}$	0	$\sqrt{3}$	0	0	0	
3	0	0	$\sqrt{3}$	0	$\sqrt{4}$	0	0	
4	0	0	0	$\sqrt{4}$	0	$\sqrt{5}$	0	
5	0	0	0	0	$\sqrt{5}$	0	$\sqrt{6}$	
6	0	0	0	0	0	$\sqrt{6}$	0	

$\langle n' | \underset{\sim}{p} | n \rangle =$

$$\frac{\hbar}{\sqrt{2b}} \times \begin{array}{c|ccccccc} & n=0 & 1 & 2 & 3 & 4 & 5 & 6, \ldots \\ \hline n'=0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\ 1 & i & 0 & -\sqrt{2}i & 0 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{2}i & 0 & -\sqrt{3}i & 0 & 0 & 0 \\ 3 & 0 & 0 & \sqrt{3}i & 0 & -\sqrt{4}i & 0 & 0 \\ 4 & 0 & 0 & 0 & \sqrt{4}i & 0 & -\sqrt{5}i & 0 \\ 5 & 0 & 0 & 0 & 0 & \sqrt{5}i & 0 & -\sqrt{6}i \\ 6 & 0 & 0 & 0 & 0 & 0 & \sqrt{6}i & 0 \\ \vdots & & & & & & & \ddots \end{array}$$

These are the representations of $\underset{\sim}{x}, \underset{\sim}{p}$ in the basis of energy eigenkets.

We know 3 bases now...

① $\langle x | \underset{\sim}{x} | x' \rangle = x\,\delta(x'-x)$ or $\underset{\sim}{x} \to x$

x $\langle x | \underset{\sim}{p} | x' \rangle = \frac{\hbar}{i}\delta'(x-x')$ or $\underset{\sim}{p} \to \frac{\hbar}{i}\frac{d}{dx}$

② $\langle p | \underset{\sim}{x} | p' \rangle = i\hbar\,\delta'(p-p')$ or $\underset{\sim}{x} \to i\hbar\frac{d}{dp}$

p $\langle p | \underset{\sim}{p} | p' \rangle = p\,\delta(p'-p)$ or $\underset{\sim}{p} \to p$

these are H-independent

③ $\underset{\sim}{H}$ dependent: when $V(\underset{\sim}{x}) = \frac{1}{2}k\underset{\sim}{x}^2$,
$\langle n'|\underset{\sim}{x}|n\rangle$, $\langle n'|\underset{\sim}{p}|n\rangle$ as previously given.

In <u>any</u> basis, $[\underset{\sim}{x}, \underset{\sim}{p}] = i\hbar$

<u>Energy</u>:

$$\underset{\sim}{x}\underset{\sim}{p} \doteq \frac{b}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 & \cdots \\ 1 & 0 & \sqrt{2} & \\ 0 & \sqrt{2} & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix} \frac{i\hbar}{\sqrt{2}b}\begin{bmatrix} 0 & -1 & 0 & \\ 1 & 0 & -\sqrt{2} & \\ 0 & \sqrt{2} & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}$$

$$\doteq \frac{i\hbar}{2}\begin{bmatrix} 1 & 0 & -\sqrt{2} & 0 \\ 0 & 1 & 0 & -\sqrt{6} \\ \sqrt{2} & 0 & 1 & \\ & \sqrt{6} & & \end{bmatrix}$$

$$\underset{\sim}{p}\underset{\sim}{x} \doteq \frac{i\hbar}{\sqrt{2}b}\begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 & -\sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} & 0 & -\sqrt{4} \\ 0 & 0 & 0 & \sqrt{4} & 0 \\ \vdots & & & & \end{bmatrix} \frac{b}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} \\ 0 & 0 & 0 & \sqrt{4} & 0 \end{bmatrix}$$

$$\doteq \frac{i\hbar}{2}\begin{bmatrix} -1 & 0 & -\sqrt{2} & 0 & 0 & \cdots \\ 0 & -1 & 0 & -\sqrt{6} & 0 & \\ \sqrt{2} & 0 & -1 & 0 & -\sqrt{12} & \\ 0 & \sqrt{6} & 0 & -1 & 0 & \\ \vdots & & \sqrt{12} & 0 & -1 & \end{bmatrix}$$

so

$\underset{\sim}{x}\underset{\sim}{p} - \underset{\sim}{p}\underset{\sim}{x} = i\hbar \cdot \underset{\sim}{I}$

The generalization of Postulate II:

old → in X basis

new → x, p (classical) $\Rightarrow \underset{\sim}{X}, \underset{\sim}{P}$ with $[\underset{\sim}{X}, \underset{\sim}{P}] = i\hbar$

$$\underbrace{\Omega(x,p)}_{\text{like}} \Rightarrow \underset{\sim}{\Omega}(\underset{\sim}{X}, \underset{\sim}{P})$$

like • Hamiltonian
• angular momentum

occasional challenge: want $\underset{\sim}{\Omega}$ to be Hermitian;
$\underset{\sim}{X}\underset{\sim}{P} \neq \underset{\sim}{P}\underset{\sim}{X}$

so xp in Ω usually replaced by $\frac{1}{2}(\underset{\sim}{X}\underset{\sim}{P} + \underset{\sim}{P}\underset{\sim}{X})$.

Generalizing to 2, 3, ... dimensions (more when >1 particles)

$[\underset{\sim}{X}, P_x] = i\hbar \quad [\underset{\sim}{Y}, P_x] = 0 \quad [\underset{\sim}{Z}, P_x] = 0$

$[\underset{\sim}{X}, P_y] = 0 \quad [\underset{\sim}{Y}, P_y] = i\hbar \quad [\underset{\sim}{Z}, P_y] = 0$

$[\underset{\sim}{X}, P_z] = 0 \quad [\underset{\sim}{Y}, P_z] = 0 \quad [\underset{\sim}{Z}, P_z] = i\hbar$

→ coordinate variables don't commute with only <u>their</u> conjugate momenta

---

$\langle n' | \underset{\sim}{\Omega}(\underset{\sim}{X}, \underset{\sim}{P}) | n \rangle$ in S.H.O.

Three ways to do this! (That we discuss)

① Represent in x-basis

$$\langle n'|\underset{\sim}{X}^3|n\rangle \doteq \int dx\, \psi_{n'}^*(x)\, x^3 \psi_n(x)$$

$$\langle n'|\underset{\sim}{X}\,\underset{\sim}{P}^2\,\underset{\sim}{X}|n\rangle \doteq \int dx\, \psi_{n'}^*(x)\, x\left(-\hbar^2 \frac{d^2}{dx^2}\right) x\, \psi_n(x)$$

② multiply matrices (did so for $\underset{\sim}{X}\underset{\sim}{P}$, $\underset{\sim}{P}\underset{\sim}{X}$)

③ expand $\underset{\sim}{X}$, $\underset{\sim}{P}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{a}^+$

specifically,

$$\underbrace{\underset{\text{odd}}{\langle 3|}\;\underset{\text{odd}}{\underset{\sim}{X}^3}\;\underset{\text{even}}{|0\rangle}}_{\text{under parity}} = \left(\frac{b}{\sqrt{2}}\right)^3 \langle 3|\underbrace{(\underset{\sim}{a}+\underset{\sim}{a}^+)(\underset{\sim}{a}+\underset{\sim}{a}^+)(\underset{\sim}{a}+\underset{\sim}{a}^+)}_{\text{look out: } [\underset{\sim}{a},\underset{\sim}{a}^+]\neq 0}|0\rangle$$

overall even, $\neq 0$

$$= \left(\frac{b}{\sqrt{2}}\right)^3 \langle 3|\left\{ \underset{\sim}{a}^3 + \underset{\sim}{a}^2\underset{\sim}{a}^+ + \underset{\sim}{a}\underset{\sim}{a}^+\underset{\sim}{a} + \underset{\sim}{a}^+\underset{\sim}{a}^2 + \underset{\sim}{a}\underset{\sim}{a}^{+2} + \underset{\sim}{a}^+\underset{\sim}{a}\underset{\sim}{a}^+ + \underset{\sim}{a}^{+2}\underset{\sim}{a} + \underset{\sim}{a}^{+3} \right\}|0\rangle$$

$$= \left(\frac{b}{\sqrt{2}}\right)^3 \langle 3|\,\underset{\sim}{a}^{+3}|0\rangle$$

$$\underset{\sim}{a}^{+3}|0\rangle = \sqrt{1}\,\underset{\sim}{a}^{+2}|1\rangle = \sqrt{1\cdot 2}\,\underset{\sim}{a}^{+}|2\rangle = \sqrt{6}\,|3\rangle$$

so $\langle 3|\underset{\sim}{X}^3|0\rangle = \dfrac{b^3}{2\sqrt{2}}\cdot\sqrt{6} = \sqrt{\dfrac{3}{4}}\, b^3$