The Classical Limit

p. 179 Chap. #6

look at the time derivative of the expectation value of an operator $\underset{\sim}{\Omega}$:

$$\frac{d}{dt}\langle\psi|\underset{\sim}{\Omega}|\psi\rangle = \frac{d\langle\psi|}{dt}\underset{\sim}{\Omega}|\psi\rangle + \langle\psi|\frac{d\underset{\sim}{\Omega}}{dt}|\psi\rangle + \langle\psi|\underset{\sim}{\Omega}\frac{d|\psi\rangle}{dt}$$

But the time derivatives of the bras + kets satisfy the Schrödinger Equation:

$$\frac{d|\psi\rangle}{dt} = \frac{\underset{\sim}{H}}{i\hbar}|\psi\rangle \rightarrow \frac{d\langle\psi|}{dt} = -\langle\psi|\frac{\underset{\sim}{H}^{+}}{i\hbar}$$

- sign is from $i\hbar$

but $\underset{\sim}{H}^{+} = \underset{\sim}{H}$ (Hamiltonian is Hermitian)

so $$\frac{d\langle\psi|}{dt} = -\frac{1}{i\hbar}\langle\psi|\underset{\sim}{H}$$

so,

$$\frac{d}{dt}\langle\underset{\sim}{\Omega}\rangle = -\frac{1}{i\hbar}\langle\psi|\underset{\sim}{H}\underset{\sim}{\Omega}|\psi\rangle + \langle\psi|\frac{d\underset{\sim}{\Omega}}{dt}|\psi\rangle + \frac{1}{i\hbar}\langle\psi|\underset{\sim}{\Omega}\underset{\sim}{H}|\psi\rangle$$

$$= \frac{1}{i\hbar}\langle\psi|(\underset{\sim}{\Omega}\underset{\sim}{H} - \underset{\sim}{H}\underset{\sim}{\Omega})|\psi\rangle + \langle\psi|\frac{d\underset{\sim}{\Omega}}{dt}|\psi\rangle$$

$$\boxed{\frac{d}{dt}\langle\underset{\sim}{\Omega}\rangle = \frac{1}{i\hbar}\langle[\underset{\sim}{\Omega},\underset{\sim}{H}]\rangle + \langle\psi|\frac{d\underset{\sim}{\Omega}}{dt}|\psi\rangle}$$

"Ehrenfest's Theorem"

Examples (all one-dimensional)

$\boxed{\underset{\sim}{\Omega} = \underset{\sim}{x}}$ and then $\frac{d\underset{\sim}{\Omega}}{dt} = 0$

$$\underset{\sim}{H} = \frac{\underset{\sim}{p}^2}{2m} + V(\underset{\sim}{x})$$

$$\frac{d}{dt}\langle \underset{\sim}{x} \rangle = \frac{1}{i\hbar}\langle [\underset{\sim}{x}, \underset{\sim}{H}] \rangle \qquad [\underset{\sim}{x}, \underset{\sim}{H}] = [\underset{\sim}{x}, (\frac{\underset{\sim}{p}^2}{2m} + V(\underset{\sim}{x}))]$$

$$= [\underset{\sim}{x}, \frac{\underset{\sim}{p}^2}{2m}] + \underbrace{\cancel{[\underset{\sim}{x}, V(\underset{\sim}{x})]}}_{0}$$

$[\underset{\sim}{x}, \underset{\sim}{p}^2] \leftarrow$ look at p.20, 1.5.10

$$= \underset{\sim}{p}\underbrace{[\underset{\sim}{x}, \underset{\sim}{p}]}_{i\hbar} + \underbrace{[\underset{\sim}{x}, \underset{\sim}{p}]}_{i\hbar}\underset{\sim}{p} = 2i\hbar \underset{\sim}{p}$$

so $$\frac{d}{dt}\langle \underset{\sim}{x} \rangle = \frac{1}{i\hbar}\langle [\underset{\sim}{x}, \underset{\sim}{H}] \rangle = \frac{1}{i\hbar}\frac{1}{2m}\langle [\underset{\sim}{x}, \underset{\sim}{p}^2] \rangle$$

$$= \frac{1}{\cancel{i\hbar}}\frac{1}{\cancel{2}m} \cdot \cancel{2i\hbar} \langle \underset{\sim}{p} \rangle$$

$$\frac{d\langle \underset{\sim}{x} \rangle}{dt} = \frac{\langle \underset{\sim}{p} \rangle}{m} \qquad \left(\frac{dx}{dt} = \frac{p}{m} \text{ is classical equation} \right)$$

$\Rightarrow$ what holds for classical variables holds for EXPECTATION VALUES in QM.

$\boxed{\underset{\sim}{\Omega} = \underset{\sim}{p}}$ $\Rightarrow$ expect, $\frac{dp}{dt} = -\frac{dV}{dx}$ (classical)

so, $\frac{d\langle p \rangle}{dt} = -\left\langle \frac{dV}{dx} \right\rangle$

$$\frac{d}{dt}\langle \underset{\sim}{p} \rangle = \frac{1}{i\hbar}\langle [\underset{\sim}{p}, \underset{\sim}{H}] \rangle = \frac{1}{i\hbar}\left(\underbrace{\langle [\underset{\sim}{p}, \frac{\underset{\sim}{p}^2}{2m}] \rangle}_{0} + \langle [\underset{\sim}{p}, V(\underset{\sim}{x})] \rangle \right)$$

$$[\underset{\sim}{P}, V(\underset{\sim}{x})]|\psi\rangle \doteq \frac{\hbar}{i}\frac{d}{dx}V(x)\psi(x) - V(x)\frac{\hbar}{i}\frac{d}{dx}\psi(x)$$

$$\doteq \frac{\hbar}{i}\frac{dV}{dx}\psi + \cancel{\frac{\hbar}{i}V\frac{d\psi}{dx}} - \cancel{V\frac{\hbar}{i}\frac{d\psi}{dx}}$$

$$= \frac{\hbar}{i}\frac{dV}{d\underset{\sim}{x}}|\psi\rangle$$

so, $\boxed{\frac{d}{dt}\langle P\rangle = \frac{1}{i\hbar}\frac{\hbar}{i}\left\langle\frac{dV}{d\underset{\sim}{x}}\right\rangle = -\left\langle\frac{dV}{dx}\right\rangle}$

as expected

$\boxed{\underset{\sim}{\Omega} = \underset{\sim}{H}}$ suppose $\underset{\sim}{H}$ = independent of time

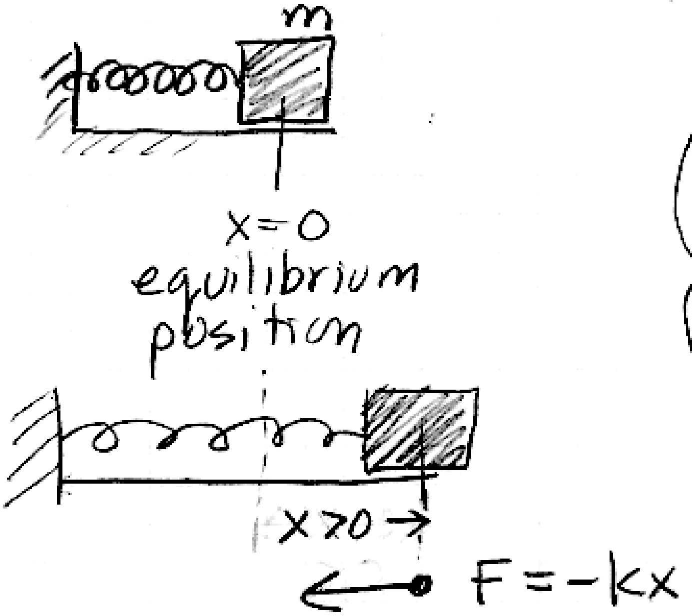
$$\frac{d}{dt}\langle\underset{\sim}{H}\rangle = \frac{1}{i\hbar}\langle[\underset{\sim}{H}, \underset{\sim}{H}]\rangle = 0$$

<u>mean</u> energy of system is conserved.

However, it appears sometimes that energy is not conserved in specific cases. Usually this kind of apparent situation results from interaction of measurement apparatus with the system, or, broadly, the focusing on a subsystem instead of the entire system.

Simple Harmonic Oscillator

Classical:



$$m\frac{d^2x}{dt^2} = -kx$$

$$x(t) = Ae^{\pm i\omega t}$$

then

$$m(\pm i\omega)^2 Ae^{\pm i\omega t} = -kAe^{\pm i\omega t}$$

$$-m\omega^2 = -k$$

$$\omega = \sqrt{\frac{k}{m}}$$

so $x(t) = \mathrm{Re}\left(Ae^{+i\omega t} + Be^{-i\omega t}\right)$

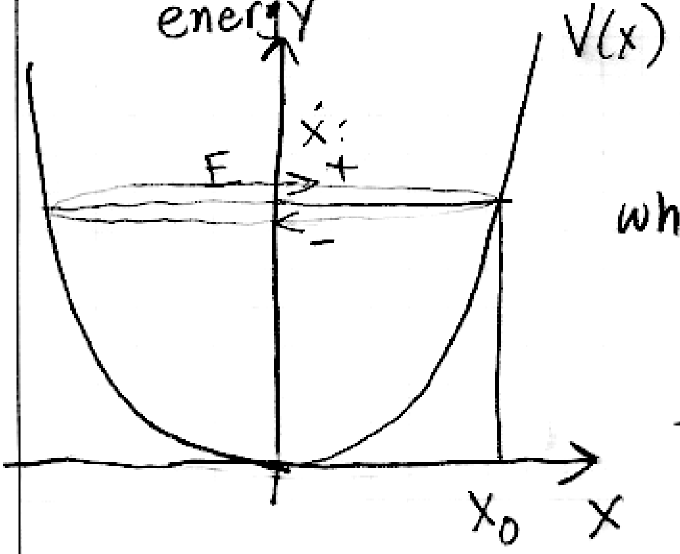
or $= \alpha\cos\omega t + \beta\sin\omega t$

or $= \alpha\cos(\omega(t-t_0))$

Classical Energy:

$$E = T + V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

where $V = -\int_0^x F\,dx' = k\int_0^x x\,dx = \frac{1}{2}kx^2$



"turning point x_0"

when E is all $V(x_0)$

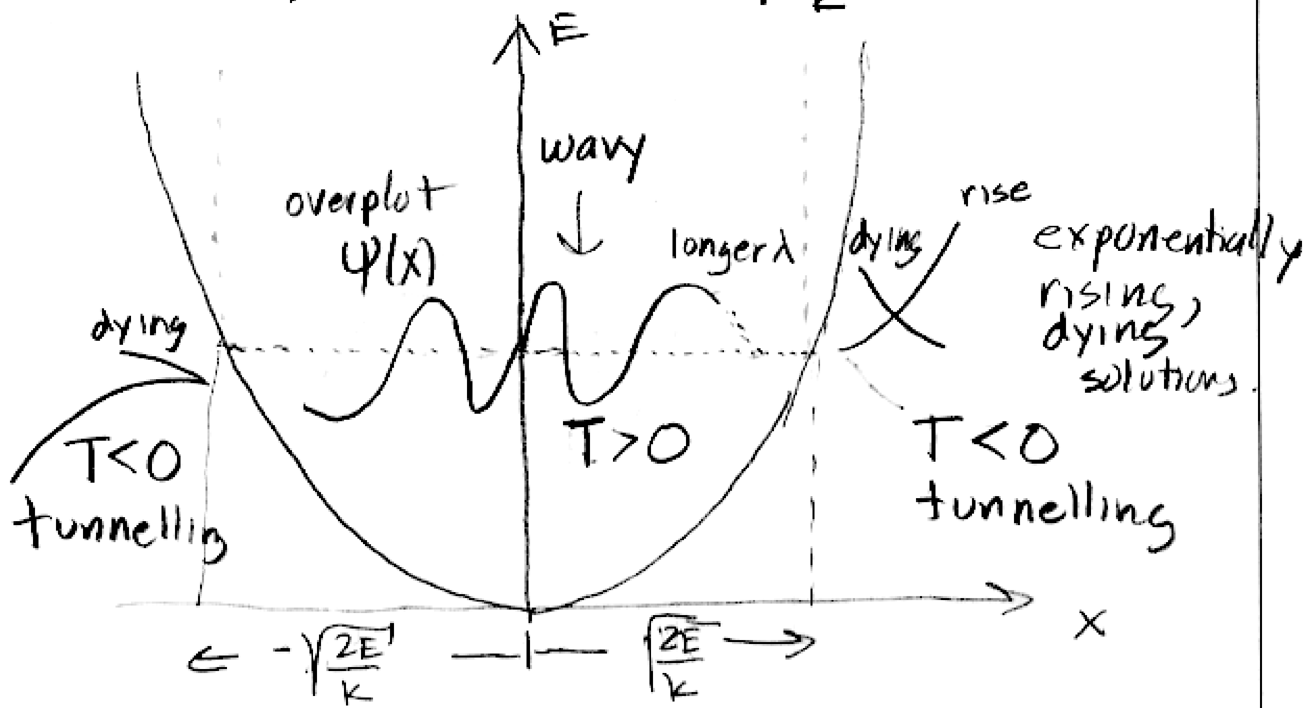
$$\frac{1}{2}kx_0^2 = E$$

$$\frac{1}{2}kx_0^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

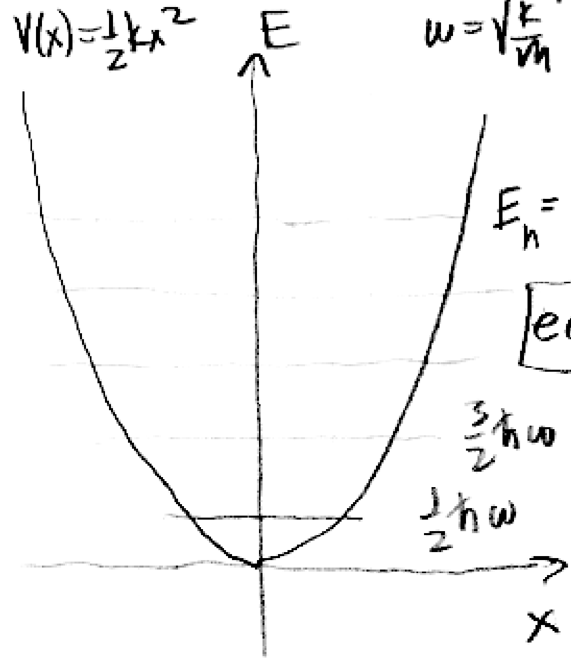
$$\dot{x} = \pm\sqrt{\frac{k}{m}}(x_0^2 - x^2)^{1/2} = \pm\omega\sqrt{x_0^2}$$

Characteristics of Quantum Oscillator

Given E, find $x_0 = \pm\sqrt{\frac{2E}{k}}$

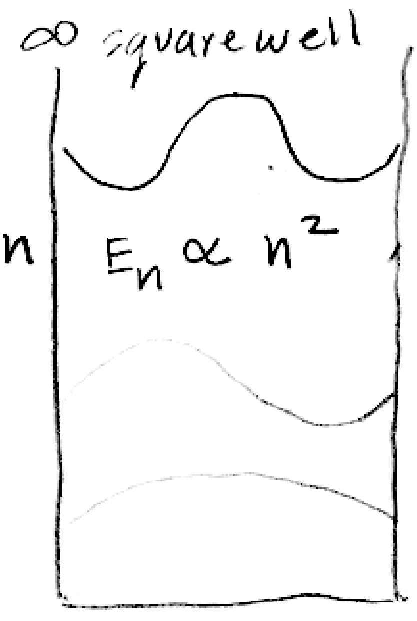


<u>Bound States</u> arise when energy is "just right" so that <u>only</u> exponentially dying solutions are needed in the tunnelling regions. Prelude of the answer.



$E_n = (n+\frac{1}{2})\hbar\omega$, $\propto n$

$\boxed{\text{equal spacing}}$



$V(x) = \frac{1}{2}kx^2 \Rightarrow$ bound state energies ≥ 0

$$E = \langle \psi | \underset{\sim}{H} | \psi \rangle$$

$$\underset{\sim}{H} = \frac{\underset{\sim}{p}^2}{2m} + \frac{1}{2}k\underset{\sim}{x}^2$$

so $E = \frac{1}{2m}\langle \psi | \underset{\sim}{p}^2 | \psi \rangle + \frac{1}{2}k \langle \psi | \underset{\sim}{x}^2 | \psi \rangle$

both $\underset{\sim}{p}$ and $\underset{\sim}{x}$ are Hermitian

so $E = \frac{1}{2m}\langle \psi | \underset{\sim}{p}^\dagger \underset{\sim}{p} | \psi \rangle + \frac{1}{2}k \langle \psi | \underset{\sim}{x}^\dagger \underset{\sim}{x} | \psi \rangle$

$$= \frac{1}{2m}\underbrace{\langle \underset{\sim}{p}\psi | \underset{\sim}{p}\psi \rangle}_{\geq 0} + \frac{1}{2}k\underbrace{\langle \underset{\sim}{x}\psi | x\psi \rangle}_{\geq 0}$$

so $E > 0$.

<u>Meaning</u>: <u>un</u>like δ- function potential ($V(x) = aV_0\,\delta(x)$) where the energy $= -\frac{1}{2}m\frac{a^2V_0^2}{\hbar^2}$. The "more space" where $T > 0$ in the oscillator pulls the energy > 0.

Eigenvalue Equation & Scale Change

(p.190)

$$\underset{\sim}{H}|E\rangle = E|E\rangle$$

$$\left(\frac{p^2}{2m} + \frac{1}{2}k\underset{\sim}{x}^2\right)|E\rangle = E|E\rangle$$

$$\underset{\sim}{H} = (\underset{\sim}{T} + \underset{\sim}{V})|E\rangle = E|E\rangle$$

represent this into coordinate space.

$$|E\rangle \doteq \langle x|E\rangle = \psi(x)$$

$$p^2|E\rangle \doteq \left(\frac{\hbar}{i}\frac{d}{dx}\right)\left(\frac{\hbar}{i}\frac{d}{dx}\right)\psi$$

$$\doteq -\hbar^2\frac{d^2\psi}{dx^2}$$

$$\frac{1}{2}k\underset{\sim}{x}^2|E\rangle \doteq \frac{1}{2}kx^2\psi(x) = \frac{1}{2}m\omega^2x^2\psi(x)$$

so: $$\boxed{\left(\underbrace{-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}}_{\underset{\sim}{T}} + \underbrace{\frac{1}{2}m\omega^2x^2}_{\underset{\sim}{V}}\right)\psi = E\psi}$$

Scale Change: let $x = by$

b has dimensions of length

dimensionless

choose b to simplify equation.