Physics 115B Midterm

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Closed Book; no calculators. For full credit, show your work and make your reasoning clear to the graders. Useful equations and information appears at the end of the test.

1. (20 pts) Consider a particle of mass m in a potential

$$V(x) = \tfrac{1}{2}m\omega^2 x^2 \qquad x > 0$$

$$= \infty \qquad x \le 0.$$

   (a) (10 pts) What is the ground state eigenvalue and *normalized* eigenfunction?

   (b) (10 pts) Use the eigenfunction for the ground state to explicitly compute the expectation value of the momentum in the ground state?

2. (15 pts) For the simple harmonic oscillator ($V(x) = \tfrac{1}{2}m\omega^2 x^2$ everywhere), compute the matrix elements $\langle n'|\boldsymbol{\Omega}|n\rangle$ for the operator:

$$\boldsymbol{\Omega} = \frac{1}{2}(\mathbf{x}\mathbf{p} + \mathbf{p}\mathbf{x}).$$

3. (15 pts) Ignore the three dimensional nature of the following system, and pretend that:

$$\mathbf{H} = \frac{\mathbf{p}^2}{2m} + \frac{1}{2}m\omega^2 \mathbf{r}^2 \quad (\mathbf{p}^2 = \mathbf{p}_x^2 + \mathbf{p}_y^2 + \mathbf{p}_z^2 \ , \ \mathbf{r}^2 = \mathbf{x}^2 + \mathbf{y}^2 + \mathbf{z}^2)$$

   corresponds to a one-dimensional problem. Assuming

$$\Delta\mathbf{p}\Delta\mathbf{r} \ge \hbar/2$$

   estimate the ground state energy.

4. (15 pts) Imagine a fictitious world in which the single-particle Hilbert space is two-dimensional. Let us denote the basis vectors by $|+\rangle$ and $|-\rangle$. Let

$$\sigma_1^{(1)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_2^{(2)} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

   be operators in the space of particle #1 and the space of particle #2, respectively; the order of the states used to represent the operators is $|+\rangle$ and then $|-\rangle$. Find the 4 by 4 matrix that represents the operator $\boldsymbol{\Omega}$:

$$\boldsymbol{\Omega} = (\sigma_1\sigma_2)^{(1)\otimes(2)} = \sigma_1^{(1)} \otimes \sigma_2^{(2)};$$

   use the order of the 4 product states as $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$.

more on the back...

5. (35 pts) Two identical particles of mass m are in a one-dimensional simple harmonic oscillator characterized by the potential $V(x) = \frac{1}{2}m\omega^2 x^2$. Energy measurement of the system yields the value $2\hbar\omega$.

   (a) (15 pts) Write down the abstract state vector(s), that is, the kets, that could describe this system. You are not told if the particles are bosons or fermions, and assume that the only degrees of freedom are orbital.

   (b) (20 pts) For all of the states found in part (a), evaluate the expectation value of $(\mathbf{x}_1 - \mathbf{x}_2)^2$, $\langle(\mathbf{x_1} - \mathbf{x_2})^2\rangle$.

The 'boldface' notation is used for operators; thus, $\mathbf{\Omega}$ is an abstract operator. In class we put a 'twiddle' under the Ω to denote that it was an operator.

The symbol $\doteq$ means 'is represented by'.

For a particle of mass m moving in the simple harmonic oscillator potential $V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2$, I assume you know the energy eigenvalues, and the abstract eigenkets $|n\rangle$ are represented by the eigenfunctions of energy:

$$\langle x|n\rangle = \psi_n(x) = \mathcal{N}_n \exp\left(-\frac{x^2}{2b^2}\right) H_n\left[\frac{x}{b}\right]$$

$$\mathcal{N}_n = \left(\frac{1}{b^2\pi 2^{2n}(n!)^2}\right)^{1/4} \quad, \quad b = \sqrt{\frac{\hbar}{m\omega}}$$

and the $H_n(y)$ are the Hermite Polynomials:

$$
\begin{array}{llll}
H_0(y) &=& 1 & \qquad H_1(y) = 2y \\
H_2(y) &=& 4y^2 - 2 & \qquad H_3(y) = 8y^3 - 12y \\
H_4(y) &=& 16y^4 - 48y^2 + 12 & \qquad H_5(y) = 32y^5 - 160y^3 + 120y
\end{array}
$$

$$I_0(\alpha) = \int_0^\infty dz e^{-\alpha z^2} = \frac{1}{2}\left[\frac{\pi}{\alpha}\right]^{1/2} \quad, \quad I_1(\alpha) = \int_0^\infty dz z e^{-\alpha z^2} = \frac{1}{2\alpha}$$

Ehrenfest's Theorem:

$$\frac{d}{dt}\langle\mathbf{\Omega}\rangle = \left(\frac{1}{i\hbar}\right)\langle[\mathbf{\Omega}, \mathbf{H}]\rangle$$

Some relationships for the algebraic solution to the simple harmonic oscillator, and other operator relationships:

$$
\begin{array}{rclcrcl}
\mathbf{a} &=& \frac{1}{\sqrt{2}}\left(\frac{\mathbf{x}}{b} + \frac{ib}{\hbar}\mathbf{p}\right) & \qquad & \mathbf{a}^\dagger &=& \frac{1}{\sqrt{2}}\left(\frac{\mathbf{x}}{b} - \frac{ib}{\hbar}\mathbf{p}\right) \\
[\mathbf{a}, \mathbf{a}^\dagger] &=& 1 & & \mathbf{H} &=& \hbar\omega(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}) \\
\mathbf{a}|n\rangle &=& \sqrt{n}|n-1\rangle & & \mathbf{a}^\dagger|n\rangle &=& \sqrt{n+1}|n+1\rangle \\
\mathbf{x} &=& \frac{b}{\sqrt{2}}\left(\mathbf{a} + \mathbf{a}^\dagger\right) & & \mathbf{p} &=& \frac{i\hbar}{b\sqrt{2}}\left(\mathbf{a}^\dagger - \mathbf{a}\right) \\
[\mathbf{\Omega}, \mathbf{\Lambda\Gamma}] &=& \mathbf{\Lambda}[\mathbf{\Omega}, \mathbf{\Gamma}] + [\mathbf{\Omega}, \mathbf{\Lambda}]\mathbf{\Gamma} & & [\mathbf{\Omega\Lambda}, \mathbf{\Gamma}] &=& \mathbf{\Omega}[\mathbf{\Lambda}, \mathbf{\Gamma}] + [\mathbf{\Omega}, \mathbf{\Gamma}]\mathbf{\Lambda}
\end{array}
$$