Here is a review of some (but not all) of the topics you should know for the midterm. These are things I think are important to know. I haven't seen the test, so there are probably some things on it that I don't cover here. Hopefully this covers most of them.

- Vector Spaces
  Review properties on Shankar page 2
  Closure under multiplication: If $|u\rangle$ and $|v\rangle \in V$, then $a|u\rangle + b|v\rangle \in V$ for any a, b. Note when $b = 0$ this takes care of scalar multiplication also.
  Inverses, identity, etc.

- Linear independence
  A set of vectors $\{|v_i\rangle\}$ is linearly independent if $a|v_1\rangle + b|v_2\rangle + \cdots = 0$ has only one solution: $a = b = \cdots = 0$.

- Gram-Schmidt procedure
  If you have a set of linearly independent vectors $|I\rangle, |II\rangle, \ldots$ you can always construct an orthonormal set of vectors as follows:

$$|1\rangle = \frac{|I\rangle}{\sqrt{\langle I|I\rangle}}$$
$$|2\rangle = \frac{|II\rangle - |1\rangle\langle 1|II\rangle}{\text{normalization constant}}$$
$$|3\rangle = \frac{|III\rangle - |1\rangle\langle 1|III\rangle - |2\rangle\langle 2|III\rangle}{\text{normalization constant}}$$
$$\cdots$$

  The normalization constants are chosen so that $\langle 2|2\rangle = 1, \langle 3|3\rangle = 1, \ldots$.

- Basis
  A basis of a vector space V is a set of vectors $\{|v_i\rangle\}$.

Any vector $|u\rangle \in V$ can be written in terms of these vectors: $|u\rangle = a|v_1\rangle = b|v_2\rangle + \ldots$ always has $a, b, \ldots$ so that the equation is satisified.

- Orthonormal (ON) basis
  An ON basis is one for which $\langle v_i|v_j\rangle = \delta_{ij}$.
- Decomposition of unity
  If $\{|v_i\rangle\}$ is an ON basis, then $\sum_i |v_i\rangle\langle v_i| = \mathbb{I}$.
- Linear Operators
  Linear operators have $\underset{\sim}{\Omega}(a|u\rangle + b|v\rangle) = a\underset{\sim}{\Omega}|u\rangle + b\underset{\sim}{\Omega}|v\rangle$.
- Operator Inverses
  The inverse of the product of operators is given by the inverses of those operators in reverse order: $(\underset{\sim}{\Omega}\underset{\sim}{\Lambda})^{-1} = \underset{\sim}{\Lambda}^{-1}\underset{\sim}{\Omega}^{-1}$.
- Commutators
  The commutator of two matrices is written $[A, B] \equiv AB - BA$. The anticommutator is written $\{A, B\} = [A, B]_+ = AB + BA$.
- Hermitian, Unitary, etc.
  An operator $\underset{\sim}{A}$ is Hermitian if $\underset{\sim}{A} = \underset{\sim}{A}^\dagger$. It is unitary if $\underset{\sim}{A}^{-1} = \underset{\sim}{A}^\dagger$ or equivalently $\underset{\sim}{A}\underset{\sim}{A}^\dagger = \mathbb{I}$.

  An operator is anti-Hermitian if $\underset{\sim}{A} = -\underset{\sim}{A}^\dagger$. An operator is anti-unitary if, among other things, $\underset{\sim}{A}(a|u\rangle) = a^*\underset{\sim}{A}|u\rangle$. Anti-Hermitian and anti-unitary operators won't show up often (if at all) in this class–in fact, I can think of only one anti-unitary operator that comes up in physics.

- Projection operators
  Defining equation: $\underset{\sim}{P}^2 = \underset{\sim}{P}$. $Tr\,\underset{\sim}{P}$ = dimensionality of subspace onto which P projects. Example: $\mathbb{I}^2 = \mathbb{I}$. The trace of an operator is the sum of the diagonal elements of its matrix representation. In N dimensions, the identity operator is a N by N matrix with N 1's on the diagonal, so $Tr\,\mathbb{I} = N$.

- Matrix elements
  Inserting a decomposition of unity twice,

$$\Omega_{ij} = \langle i|\underset{\sim}{\Omega}|j\rangle$$

$$\underset{\sim}{\Omega} = \sum_{ij} |i\rangle \Omega_{ij} \langle j|$$

  For a vector, the components are given by

$$v_i = \langle i|v\rangle$$

$$|v\rangle = \sum_i |i\rangle \underbrace{\langle i|v\rangle}_{v_i} = \sum_i v_i |i\rangle$$

- Change of basis
  A change of basis from one ON basis (the "unprimed" basis $\{|i\rangle\}$) to another basis (the "primed" basis $\{|i'\rangle\}$) transforms operators and vectors as follows (inserting decompositions of $\mathbb{I}$),

$$\underbrace{\langle i'|\underset{\sim}{A}|j'\rangle}_{A'_{i'j'}} = \sum_{ij} \underbrace{\langle i'|i\rangle}_{(U^\dagger)_{i'i}} \underbrace{\langle i|\underset{\sim}{A}|j\rangle}_{A_{ij}} \underbrace{\langle j|j'\rangle}_{(U)_{jj'}}$$

$$\underbrace{\langle i'|v\rangle}_{v'_i} = \sum_i \underbrace{\langle i'|i\rangle}_{(U^\dagger)_{i'i}} \underbrace{\langle i|v\rangle}_{v_i}$$

Note that $U_{jj'}$ is a matrix for which the j'^{th} basis vector goes in j'^{th} column.

- Eigenvectors, eigenvalues
  If

$$\underset{\sim}{A}|v\rangle = a|v\rangle \qquad |v\rangle \neq 0$$

then $|v\rangle$ is an eigenvector of $\underset{\sim}{A}$ with eigenvalue a.

- Determining eigenvalues Solve the equation

$$det\Big(A - a\mathbb{I}\Big) = 0$$

where A is a matrix representation of $\underset{\sim}{A}$. The left hand side ends up being a polynomial called the "characteristic polynomial" of the operator, and the equation is called the "characteristic equation" of the operator. For an N by N matrix A, the polynomial is an N^{th}-order polynomial, and so the equation has N solutions. They need not be distinct – one or more of the eigenvalues can be the same number. If that happens, that eigenvalue is called "degenerate."

- Eigenvectors
  Once you have the eigenvalues, solve

$$A \begin{bmatrix} \alpha \\ \beta \\ \ldots \end{bmatrix} = a \begin{bmatrix} \alpha \\ \beta \\ \ldots \end{bmatrix}$$

for each eigenvalue to get the associated eigenvector $(\alpha, \beta, ...)$. If the eigenvalue is nondegenerate, you'll have N unknowns $\alpha, \beta, ...$ and $N-1$ equations $\implies$

one-parameter family of eigenvectors. Impose normalization condiition $\alpha^*\alpha + \beta^*\beta\dot{..} = 1$ to fix the final free parameter.
If the eigenvalue is m-fold degenerate (m of the eigenvalues are the same) then you get N free parameters and $N-(1+m)$ equations, and thus an m-parameter family of eigenvectors. Example: suppose you get the eigenvector

$$\begin{aligned} |v\rangle &= \begin{bmatrix} \alpha \\ \beta \\ -\beta \end{bmatrix} \\ &= \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \end{aligned}$$

You can split it up into two or more "basis" vectors that "span the degenerate subspace"–in the above example, any eigenvector of that eigenvalue can be written as a linear combination of the two vectors with combination coefficients α and β.

- Diagonalization
If $\{|i\rangle\}$ are the normalized eigenvectors of A, you can represent A in that "eigenbasis", and if the eigenvectors are normalized, the new matrix representation will be a diagonal matrix with the eigenvalues as the diagonal elements. As discussed for changes of basis, U is constructed

$$U = \begin{bmatrix} |1\rangle & |2\rangle & \dots \end{bmatrix}$$

If the eigenvectors are not normalized, you'll still get a diagonal matrix, but the diagonal elements will not be the eigenvalues of A.

- Simultaneous Diagonalization
  Suppose we have a matrix B that commutes with A: $[A, B] = 0$. Then the ON basis that diagonalizes A is the same ON basis that diagonalizes B–the eigenvectors of B are the same as the (orthonormal) eigenvectors of A, but with different eigenvalues ($B|i\rangle = b|i\rangle$). They diagonalize B into a matrix with B's eigenvalues on the diagonal.

  One of the reasons one cares about this is illustrated as follows. Suppose you have a 1000 by 1000 matrix B. The characteristic equation is a 1000th-order polynomial. For 2nd order polynomials, the quadratic equation can solve the characteristic equation; for 3rd and 4th order polynomials we also have equations. But for higher-order polynomials there is no general way of finding the roots, and so finding the eigenvalues would be very hard. But, if you can find an A that commutes with B, you can find the eigenvalues and eigenvectors of A instead of solving the characteristic equation for B. If you can find an A for which diagaonlization is very easy, then all you have to do is matrix multiplication to diagonalize B and find its eigenvalues. It saves a lot of work.

- Delta functions
  Suppose you have an interval γ (e.g. $\gamma = (-\infty, \infty)$.

Then the defining equation of a delta function is

$$\int_\gamma f(x)\delta(x)dx = f(0)$$

if $0 \in \gamma$ and the result is zero if zero is not in the interval. Also, integrating by substituting $u = g(x)$,

$$\int_\gamma f(x)\delta(g(x))dx = \int_\gamma f(x(u))\delta(u)\frac{du}{\frac{dg(x(u))}{dx}} = \sum_i \frac{f(x_i)}{|g'(x_i)|}$$

where x_i is a solution of $g(x_i) = 0$ and $x_i \in \gamma$. Why the absolute value sign is required is a homework problem for Monday. Finally, integrating by parts,

$$\int_\gamma f(x)\frac{d\delta(x)}{dx}dx = f(x)\delta(x)\bigg|_{\partial\gamma} - \int_\gamma \frac{df}{dx}\delta(x)dx$$

where $\partial\gamma$ is the boundary of the interval γ (e.g. if $\gamma = (-1, 1)$, $f(x)\delta(x)|_{\partial\gamma} = f(x)\delta(x)|_{-1}^{1}$. Since $\delta(x)$ is zero everywhere except $x = 0$, the first term is zero as long as $0 \in \gamma$ (and not on the boundary) and so

$$\int_\gamma f(x)\frac{d\delta(x)}{dx}dx = -\int_\gamma \frac{df}{dx}\delta(x)dx$$