1, (a) $|\langle\psi|\omega\rangle|^2 = \left|\frac{1}{\sqrt2}(-1\ \ 1)\frac{1}{\sqrt5}\begin{pmatrix}2\\1\end{pmatrix}\right|^2$

$\omega = E_1 = +\frac{5\hbar\pi}{2}$

$$\boxed{P\left(+5\frac{\hbar\pi}{2}\right) = \left|\frac{1}{\sqrt{10}}(-2+1)\right|^2 = \frac{1}{10}}$$

$$P\left(-5\frac{\hbar\pi}{2}\right) = \left|\frac{1}{\sqrt2}(-1\ \ 1)\frac{1}{\sqrt5}\begin{pmatrix}-1\\2\end{pmatrix}\right|^2$$

$$\boxed{P\left(-5\frac{\hbar\pi}{2}\right) = \left|\frac{1}{\sqrt{10}}(1+2)\right|^2 = \frac{9}{10}}$$ (or, $= 1-\frac{1}{10} = \frac{9}{10}$

(b) $$\boxed{|+5\rangle\langle+5| \doteq \frac{1}{5}\begin{pmatrix}2\\1\end{pmatrix}(2\ \ 1) = \frac{1}{5}\begin{pmatrix}4 & 2\\2 & 1\end{pmatrix}}$$

$$\boxed{|-5\rangle\langle-5| \doteq \frac{1}{5}\begin{pmatrix}-1\\2\end{pmatrix}(-1\ \ 2) = \frac{1}{5}\begin{pmatrix}1 & -2\\-2 & 4\end{pmatrix}}$$

or, $|-5\rangle\langle-5| = 1 - |+5\rangle\langle+5|$

$$= \frac{1}{5}\begin{pmatrix}5 & 0\\0 & 5\end{pmatrix} - \frac{1}{5}\begin{pmatrix}4 & 2\\2 & 1\end{pmatrix} = \frac{1}{5}\begin{pmatrix}1 & -2\\-2 & 4\end{pmatrix}$$

(c) $$\underset{\sim}{U}(t,0) \doteq |+5\rangle\langle+5|e^{-\frac{iE_1t}{\hbar}} + |-5\rangle\langle-5|e^{-\frac{iE_2t}{\hbar}}$$

$$\doteq \frac{1}{5}\begin{pmatrix}4 & 2\\2 & 1\end{pmatrix}e^{-\frac{i5\pi t}{2}} + \frac{1}{5}\begin{pmatrix}1 & -2\\-2 & 4\end{pmatrix}e^{\frac{i5\pi t}{2}}$$

$$\boxed{\underset{\sim}{U}(t,0) \doteq \frac{1}{5}\begin{pmatrix}4e^{-\frac{i5\pi t}{2}} + e^{\frac{i5\pi t}{2}} & 2\left(e^{-\frac{i5\pi t}{2}} - e^{\frac{i5\pi t}{2}}\right)\\ 2\left(e^{-\frac{i5\pi t}{2}} - e^{\frac{i5\pi t}{2}}\right) & e^{-\frac{i5\pi t}{2}} + 4e^{\frac{i5\pi t}{2}}\end{pmatrix}}$$

(d) $\underset{\sim}{U}(\tfrac{1}{5},0) = \frac{1}{5}\begin{pmatrix} 4e^{-\frac{i\pi}{2}} + e^{\frac{i\pi}{2}} & 2(e^{-\frac{i\pi}{2}} - e^{\frac{i\pi}{2}}) \\ 2(e^{-\frac{i\pi}{2}} - e^{\frac{i\pi}{2}}) & e^{-\frac{i\pi}{2}} + 4e^{\frac{i\pi}{2}} \end{pmatrix}$

$e^{-\frac{i\pi}{2}} = -i \qquad e^{\frac{i\pi}{2}} = i$

$$\underset{\sim}{U}(\tfrac{1}{5},0) = \frac{1}{5}\begin{pmatrix} -4i+i & 2(-i-i) \\ 2(-i-i) & -i+4i \end{pmatrix}$$

$$= \frac{-i}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$$

$$|\psi(\tfrac{1}{5})\rangle = \underset{\sim}{U}(\tfrac{1}{5},0)|\psi(0)\rangle$$

$$\doteq -\frac{i}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{-i}{5\sqrt{2}}\begin{pmatrix} -3+4 \\ -4-3 \end{pmatrix}$$

$$|\psi(\tfrac{1}{5})\rangle \doteq \frac{-i}{5\sqrt{2}}\begin{pmatrix} 1 \\ -7 \end{pmatrix}$$

$$\boxed{P(+5\tfrac{\hbar\pi}{2}) = \left|\frac{i}{5\sqrt{2}}(1\ -7)\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right|^2 = \frac{1}{250}(2-7)^2 = \frac{1}{10}}$$

$$\boxed{P(-5\tfrac{\hbar\pi}{2}) = \left|\frac{i}{5\sqrt{2}}(1\ -7)\frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix}\right|^2 = \frac{1}{250}(-1-14)^2 = \frac{225}{250} = \frac{9}{10}}$$

2. (a) $$\int_{-\infty}^{\infty} dx\,|\psi(x)|^2 = |A|^2\int_{-\infty}^{0} dx\, e^{2Kx} + |A|^2\int_{0}^{\infty} dx\, e^{-2Kx}$$

$$= 2|A|^2\int_{0}^{\infty} dx\, e^{-2Kx} = \frac{2|A|^2}{2K} = \frac{|A|^2}{K} = 1$$

$$|A| = \sqrt{K}$$

(b) $\boxed{\langle \underset{\sim}{x} \rangle = \int_{-\infty}^{\infty} dx \, \underset{\substack{\uparrow \\ \text{odd}}}{x} \, \underset{\substack{\uparrow \\ \text{even}}}{|\psi(x)|^2} = 0}$

$$\langle \underset{\sim}{x}^2 \rangle = \int_{-\infty}^{\infty} dx \, x^2 |\psi(x)|^2$$

$$= 2K \int_{-\infty}^{\infty} dx \, x^2 e^{-2Kx}$$

$$= 2K \cdot \frac{2}{2^3 K^3}$$

$$\boxed{\langle \underset{\sim}{x}^2 \rangle = \frac{1}{2K^2}}$$

$$\int_0^\infty dx \, e^{-\alpha x} = \frac{1}{\alpha}$$

$$-\frac{d}{d\alpha}\left[\int_0^\infty dx \, e^{-\alpha x}\right] = \frac{1}{\alpha^2}$$

or $\int_0^\infty dx \, x e^{-\alpha x} = \frac{1}{\alpha^2}$

and $\int_0^\infty dx \, x^2 e^{-\alpha x} = \frac{2}{\alpha^3}$

(c) $\langle V(x) \rangle = \int_{-\infty}^{\infty} dx \left(-aV_0 \delta(x)\right) |\psi(x)|^2$

$$\boxed{\langle V(x) \rangle = -aV_0 |\psi(0)|^2 = -aV_0 K}$$

(d) $\langle p \rangle = \int_{-\infty}^{\infty} dx \, \psi^*(x) \frac{\hbar}{i} \frac{d}{dx} \psi(x)$

$$= K \int_{-\infty}^{0} dx \, e^{+Kx} \frac{\hbar}{i} \frac{d}{dx} e^{+Kx} + K \int_0^\infty dx \, e^{-Kx} \frac{\hbar}{i} \frac{d}{dx} e^{-Kx}$$

$$\boxed{\langle p \rangle = \frac{\hbar}{i} K^2 \left[\int_{-\infty}^{0} dx \, e^{2Kx} - \int_0^\infty dx \, e^{-2Kx}\right] = 0}$$

$$\langle p^2\rangle = \int_{-\infty}^{\infty} dx\, \psi^*(x)\left(-\hbar^2 \frac{d^2}{dx^2}\right)\psi(x)$$

this turns out to be subtle.....

(i) "Too straight-forward"

$$= -\hbar^2 K\left[\int_{-\infty}^{0} dx\, e^{Kx}\frac{d^2}{dx^2}e^{Kx} + \int_{0}^{\infty} dx\, e^{-Kx}\frac{d^2}{dx^2}e^{-Kx}\right]$$

$$= -\hbar^2 K^3\left[2\cdot\int_0^{\infty} dx\, e^{-2Kx} = \frac{2}{2K}\right]$$

$$\langle p^2\rangle = -\hbar^2 K^2 < 0$$ WRONG.. (but not very wrong)

why? because

(ii) $$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - aV_0\,\delta(x)\psi = E\psi$$

so $$-\hbar^2\frac{d^2\psi}{dx^2} = 2m\big(E + aV_0\,\delta(x)\big)\psi$$

$$-\frac{ma^2V_0^2}{2\hbar^2} = -\frac{\hbar^2}{2m}\times K^2$$

$$2mE = -\hbar^2K^2$$

computation above gives this term.

But the δ-function contributes:

$$2mK\int_{-\infty}^{\infty} dx\, e^{\pm Kx} a V_0\, \delta(x)\, e^{\mp Kx} = 2maV_0 K$$

$$K = \frac{mV_0 a}{\hbar^2}, \quad \text{so} \quad = 2\hbar^2 K^2$$

thus,

$$\langle p^2 \rangle = \underbrace{-\hbar^2 K^2}_{\text{from energy or direct integration}} + \underbrace{2\hbar^2 K^2}_{\text{from } \delta\text{-function}}$$

$$\boxed{\langle p^2 \rangle = +\hbar^2 K^2}$$

iii) since $E = -\dfrac{ma^2V_0^2}{2\hbar^2}$

and $\langle V(x) \rangle = -aV_0 K = -\dfrac{ma^2V_0^2}{\hbar^2}$

$$\langle T \rangle = E - \langle V(x) \rangle$$

$$\frac{\langle p^2 \rangle}{2m} = -\frac{ma^2V_0^2}{2\hbar^2} + \frac{ma^2V_0^2}{\hbar^2} = +\frac{ma^2V_0^2}{2\hbar^2}$$

$$\langle p^2 \rangle = \frac{m^2a^2V_0^2}{\hbar^2} = +\hbar^2K^2$$

(e) $\frac{\langle p^2\rangle}{2m} = \frac{\hbar^2 K^2}{2m} = \frac{m a^2 V_0^{\,2}}{2\hbar^2}$

(f). $\langle \underset{\sim}{x}\rangle = \langle p\rangle = 0$

so $\Delta x = \sqrt{\langle \underset{\sim}{x}^2\rangle} = \frac{1}{\sqrt{2}}\frac{1}{K}$

$\Delta p = \sqrt{\langle p^2\rangle} = \hbar K$ (ignore δ-function, $= \pm i\hbar K$)

$\Delta x\,\Delta p = \frac{1}{\sqrt{2}}\hbar \;\left(> \frac{1}{2}\hbar\right)$ $\left[\pm\frac{i}{\sqrt{2}}\hbar\right]$

(g) $\langle p|\psi\rangle = \int dx \langle p|x\rangle\langle x|\psi\rangle$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx\, e^{\frac{ipx}{\hbar}} \psi(x)$$

$$= \frac{\sqrt{K}}{\sqrt{2\pi\hbar}} \left[\int_{-\infty}^{0} dx\, e^{\frac{ipx}{\hbar}} e^{Kx} + \int_{0}^{\infty} dx\, e^{+\frac{ipx}{\hbar}} e^{-Kx} \right]$$

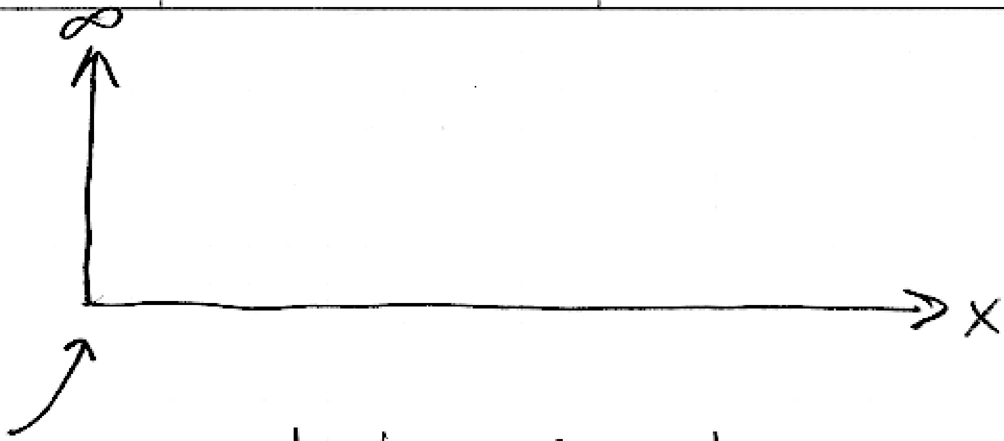
$$= \sqrt{\frac{K}{2\pi\hbar}} \left[\left. \frac{e^{\frac{ipx}{\hbar} + Kx}}{\frac{ip}{\hbar} + K} \right|_{-\infty}^{0} + \left. \frac{e^{\frac{ipx}{\hbar} - Kx}}{\frac{ip}{\hbar} - K} \right|_{0}^{\infty} \right]$$

$$= \sqrt{\frac{K}{2\pi\hbar}} \left[\frac{1}{\frac{ip}{\hbar} + K} - \frac{1}{\frac{ip}{\hbar} - K} \right]$$

$$= \sqrt{\frac{K}{2\pi\hbar}} \left[\frac{\cancel{\frac{ip}{\hbar}} - K - \cancel{\frac{ip}{\hbar}} - K}{\left(\frac{ip}{\hbar} + K\right)\left(\frac{ip}{\hbar} - K\right)} \right]$$

$$\boxed{\langle p|\psi\rangle = \sqrt{\frac{K}{2\pi\hbar}} \; \frac{2K}{K^2 + \left(\frac{p}{\hbar}\right)^2}}$$

3. (a)



functions must be 0 at $x=0$.

otherwise, $V(x)=0$ for $x>0$.

so,
$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi(x)$$

$$\psi = A_{\pm}e^{\pm ikx} \qquad k=\sqrt{\frac{2mE}{\hbar^2}}$$

$$E \geq 0$$

(states with $E<0$ not normalizable, with boundary condition)

$$\psi(0) = 0 = A_+e^{+ik\cancel{x}^{0}} + A_-e^{-ik\cancel{x}^{0}}$$

$$0 = A_+ + A_-$$

$$\boxed{A_- = -A_+}$$

$$\boxed{\begin{gathered}\psi(x) = 2iA_+\sin(kx) = \alpha\sin(kx) \\ \text{with } k=\sqrt{\frac{2mE}{\hbar^2}} \quad E>0\end{gathered}}$$

(b) $\psi(x) = \alpha \sin(kx)$

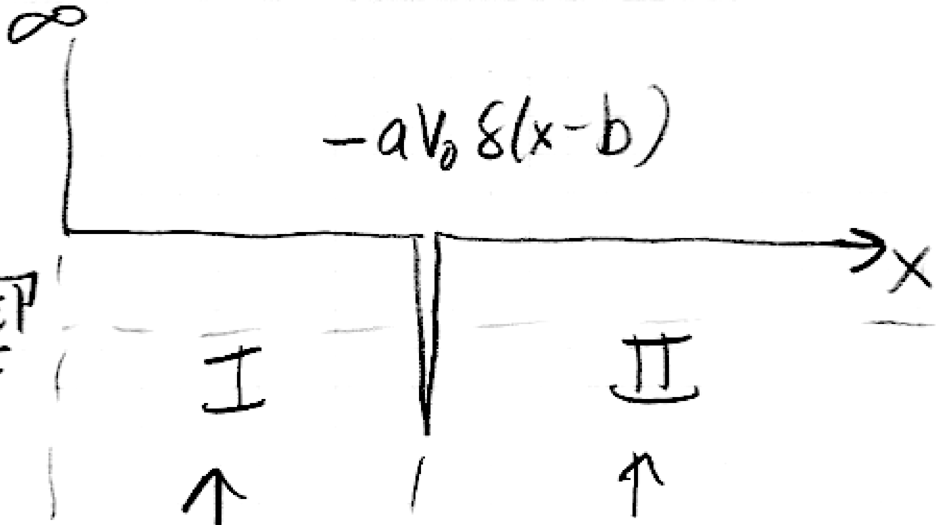
$\psi^*(x) = \alpha^* \sin(kx)$

$\frac{d\psi(x)}{dx} = \alpha k \cos(kx)$ $\qquad$ $\frac{d\psi^*}{dx} = \alpha^* k \cos kx$

$\psi^* \frac{d\psi}{dx} - \psi \frac{d\psi^*}{dx} = \alpha^* \alpha k \sin(kx)\cos(kx)$
$- \alpha \alpha^* k \sin(kx)\cos(kx)$

$$\boxed{j = 0}$$

4. ⓐ



$\kappa = \sqrt{\dfrac{2m|E|}{\hbar^2}}$

$\psi_I = Ae^{\kappa x} + Be^{-\kappa x}$ $\qquad$ $\psi_{II} = Ce^{-\kappa x}$

$$\psi(0) = 0 = A + B \quad \Rightarrow \quad B = -A$$

$$\psi_I(b) = \psi_{II}(b) \Rightarrow \quad Ae^{\kappa b} + Be^{-\kappa b} = Ce^{-\kappa b}$$

or $\quad A(e^{\kappa b} - e^{-\kappa b}) = Ce^{-\kappa b}$. $\qquad$ #1

for $\psi'(b)$ integrate across the δ-function.

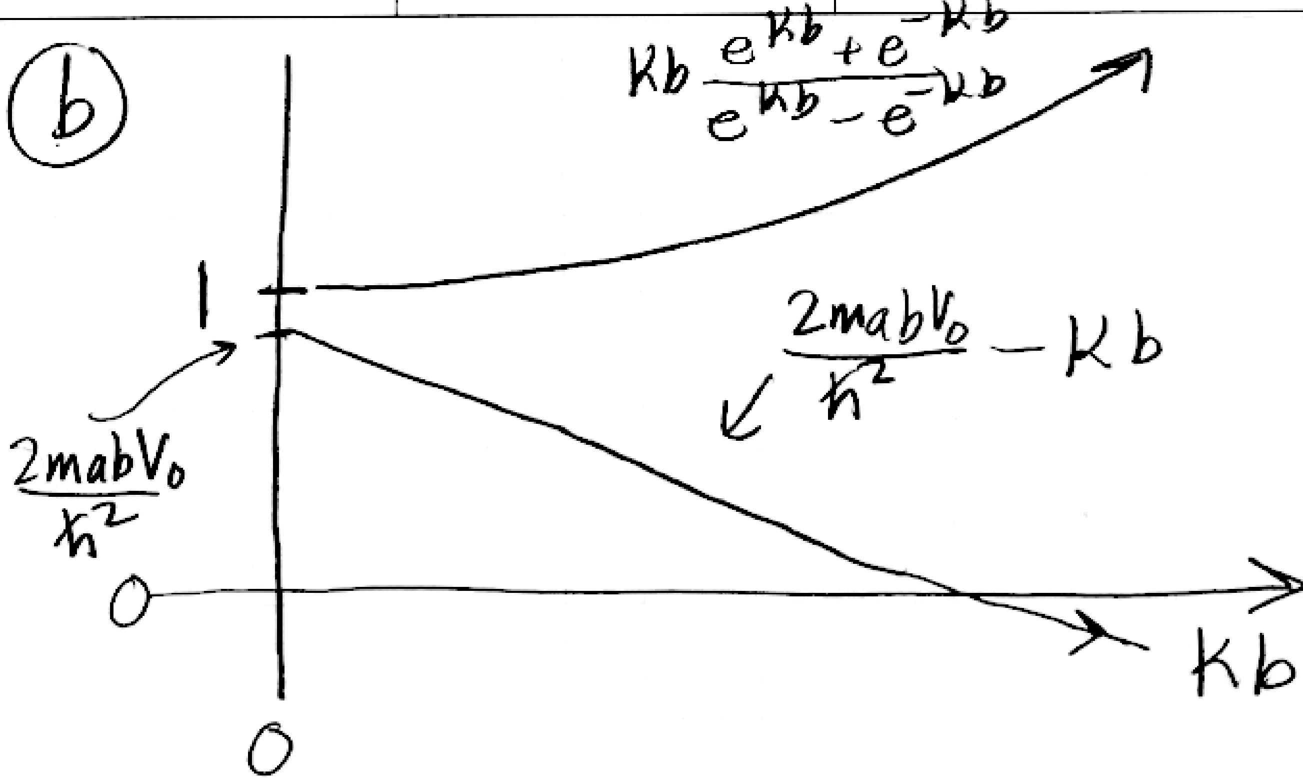
$$\psi'_{II}(b) - \psi'_I(b) = -\frac{2maV_0}{\hbar^2}\psi_I(b)$$

$$-\kappa Ce^{-\kappa b} - \kappa A(e^{\kappa b} + e^{-\kappa b}) = -\frac{2maV_0}{\hbar^2}Ce^{-\kappa b}$$

or $\quad A\kappa(e^{\kappa b} + e^{-\kappa b}) = \left(\dfrac{2maV_0}{\hbar^2} - \kappa\right)Ce^{-\kappa b}$ $\qquad$ #2

b. $\dfrac{\#2}{\#1} = \boxed{\kappa b\,\dfrac{e^{\kappa b} + e^{-\kappa b}}{e^{\kappa b} - e^{-\kappa b}} = \left(\dfrac{2mabV_0}{\hbar^2} - \kappa b\right)}$

(b)



no bound states: $\frac{2mabV_0}{\hbar^2} < 1$

1 bound state: $\frac{2mabV_0}{\hbar^2} \geq 1$