Physics 115A Final

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Tuesday, Mar. 18, 2003

Closed Book; calculators allowed. For full credit, show your work and make your reasoning clear to the graders.

Some equations and numbers appear at the end of the exam.

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1. (20 pts) The Hamiltonian, $\mathbf{H}$ for a two-state physical system is:

$$\mathbf{H} \doteq \frac{\hbar\pi}{2}\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

The eigenvalues and eigenkets (represented in the same basis as the Hamiltonian above) for this system are:

$$E_1 = +\tfrac{5\hbar\pi}{2}, \quad |+5\rangle \doteq \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$E_2 = -\tfrac{5\hbar\pi}{2}, \quad |-5\rangle \doteq \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

(a) At time $t = 0$, the system is in the state $|\psi\rangle$, where

$$|\psi\rangle \doteq \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Evaluate the probability of measuring the eigenvalue E_1, and the probability of measuring the eigenvalue E_2.

(b) Find the matrices that represent the operators $|+5\rangle\langle+5|$ and $|-5\rangle\langle-5|$, in the same basis as the Hamiltonian above.

(c) Find the matrix that represents the propagator, $\mathbf{U}(t,0)$, in the same basis as the Hamiltonian above.

(d) At the time $t = 1/5$, evaluate the probability of measuring the eigenvalue E_1, and the probability of measuring the eigenvalue E_2.

2. (30 pts) The potential $V(x) = -aV_0\delta(x)$ has one bound state, with energy $E = -ma^2V_0^2/2\hbar^2$, and a wave function:

$$\begin{aligned} \psi(x) &= Ae^{+\kappa x} \quad \text{for } x < 0 \\ \psi(x) &= Ae^{-\kappa x} \quad \text{for } x > 0 \end{aligned}$$

where A is a normalization constant, and $\kappa = \sqrt{-2mE/\hbar^2}$.

(a) Evaluate the normalization constant A.

(b) Evaluate the expectation values of $\mathbf{x}$ and $\mathbf{x^2}$.

(c) Evaluate the expectation value of the potential energy.

(d) Evaluate the expectation values of $\mathbf{p}$ and $\mathbf{p^2}$.

(e) Evaluate the expectation value of the kinetic energy, which is the expectation value of $\mathbf{p}^2/2m$.

(f) Evaluate the product $\Delta x \Delta p$.

(g) Find the representation of this wave function in momentum space. Recall that $\langle x|p\rangle = [\exp{(ipx/\hbar)}]/\sqrt{2\pi\hbar}$.

3. (20 pts) Consider the potential

$$\begin{aligned} V(x) &= \infty \qquad \text{for } x \leq 0 \\ V(x) &= 0 \qquad \text{for } x > 0 \end{aligned}$$

(a) Describe the eigenstates of the Hamiltonian, and the eigenvalues of the Hamiltonian (aka, the allowed energies). You can neglect the normalization of the eigenstates.

(b) If a particle is in an eigenstate of this Hamiltonian, can the probability current density:

$$j = \frac{\hbar}{2mi}(\psi^\star \frac{d\psi}{dx} - \psi \frac{d\psi^\star}{dx})$$

be non-zero?

4. (30 pts) Consider the potential that consists of the superposition of that from the previous problem, and a delta function at the position $x = b$, where $b > 0$:

$$\begin{aligned} V(x) &= \infty \qquad & \text{for } x \leq 0 \\ V(x) &= -aV_0\delta(x-b) \qquad & \text{for } x > 0. \end{aligned}$$

(a) Derive an equation that implicitly determines the energy(ies) of any bound state(s).

(b) What (if any) equation must a, b, and V_0 satisfy for a bound state to exist?

The 'boldface' notation is used for operators; thus, $\mathbf{\Omega}$ is an abstract operator. In class we put a 'twiddle' under the Ω to denote that it was an operator. The symbol $\doteq$ means 'is represented by'.

Some equations:

$$\begin{aligned} \langle x|\mathbf{p}|\psi\rangle &= \frac{\hbar}{i}\frac{d\psi}{dx} \\ \frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) &= E\psi(x) \\ \mathbf{U}(t,0) &= \sum_{i=1}^{n} |E_i\rangle e^{-iE_it/\hbar}\langle E_i| \\ k &= \sqrt{\tfrac{2mE}{\hbar^2}} \\ \kappa &= \sqrt{\tfrac{2m(V_0-E)}{\hbar^2}} \\ \kappa a &= ka\tan(ka) \quad \text{even bound states} \\ \kappa a &= -ka\cot(ka) \quad \text{odd bound states} \\ (ka)^2 + (\kappa a)^2 &= \tfrac{2mV_0a^2}{\hbar^2} \\ m_e &= 9.1 \times 10^{-31}\,\text{kg} \\ m_p &= 1.7 \times 10^{-27}\,\text{kg} \\ 1\,\text{eV} &= 1.6 \times 10^{-19}\ \text{Joules} \\ \hbar &= 1.05 \times 10^{-34}\ \text{Joule-seconds} \end{aligned}$$