often, to eliminate the uninteresting normalization, we take the ratio:

$$L \times \frac{\psi'(L)}{\psi(L)} = kL \cot kL = \frac{-\kappa L\left(Be^{-\kappa L} - Ce^{\kappa L}\right)}{Be^{-\kappa L} + Ce^{\kappa L}}$$

(#2) For a localized state, $C=0$.
Why? C multiplies a <u>rising</u> exponential; if $C \neq 0$, then $|\psi(x)|^2 \to \infty$ when $x \to \infty$, and the particle's probability is not local,

For a localized state:

$$kL \cot kL = -\kappa L$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$k = \sqrt{\frac{2mE}{\hbar^2}} \qquad\qquad \kappa = \sqrt{\frac{2m}{\hbar^2}(V_0 - E)}$$

so, note: $k^2 + \kappa^2 = \frac{2m}{\hbar^2}(E + V_0 - E) = \frac{2mV_0}{\hbar^2}$

$\uparrow$
constant.

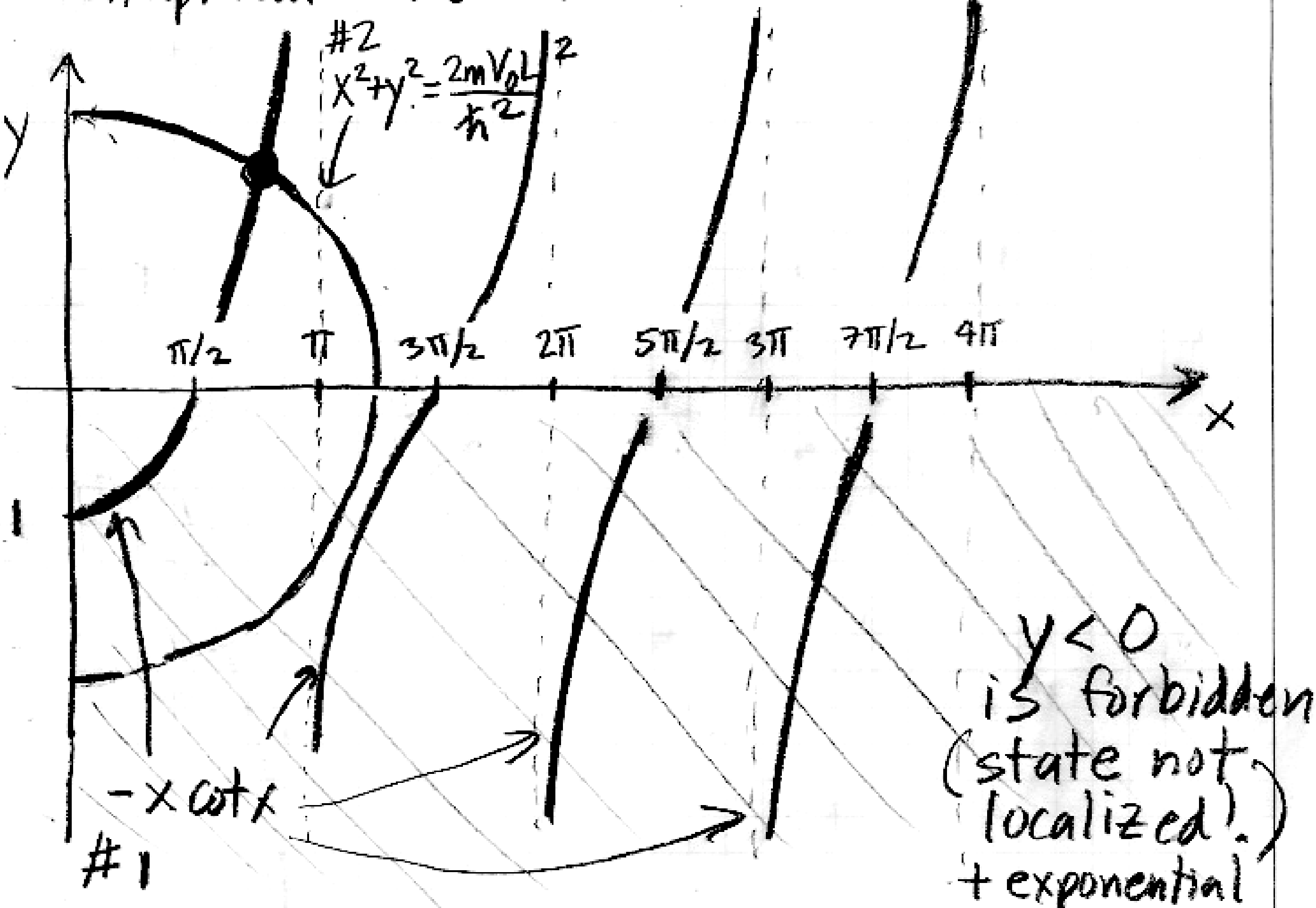
or $$(kL)^2 + (\kappa L)^2 = \frac{2mV_0L^2}{\hbar^2}$$

- Two equations in two unknowns, but non-linear (and transcendental)

- denote $y \equiv \kappa L > 0 \quad x = kL > 0$

#1: $y = -x \cot x$ #2: $x^2 + y^2 = \frac{2mV_0L^2}{\hbar^2}$

• Graphical Solution:



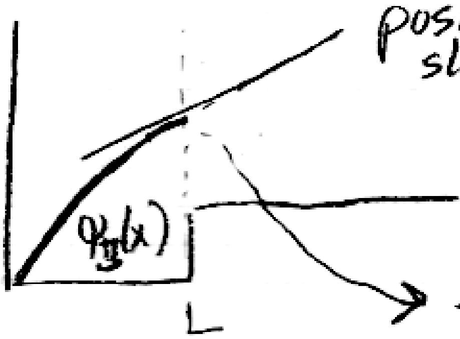
The intersections of the two curves determine the allowed (or quantized) values of k and κ.

Note that as $V_0 \to 0$, one reaches a situation where there is not even one bound state (!). No bound states when

$$x^2+y^2 < \left(\frac{\pi}{2}\right)^2 \quad \text{or}$$

$$\text{or} \quad \boxed{\frac{2mV_0L^2}{\hbar^2} < \frac{\pi^2}{4}} \quad V_0 < \frac{\hbar^2\pi^2}{8mL^2}$$

Physical Picture of this:



positive slope at $x = L$ (need positive exponential to match in $x > L$ region)

$V_0 < \frac{\hbar^2 \pi^2}{8mL^2}$

Then for $E = V_0$; $k = \sqrt{\frac{2mV_0}{\hbar^2}}$

$\psi(x) \propto \sin kx$

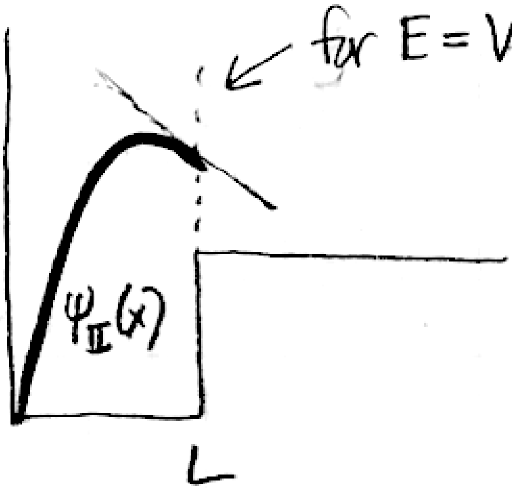
$\psi'(L) \propto \cos kL > 0$ since

$kL < \pi/2$

$\sqrt{\frac{2mV_0}{\hbar^2}}\, L < \pi/2$

$V_0 < \frac{\hbar^2 \pi^2}{8mL^2}$

Positive slope: need $C \neq 0$, $Ce^{+\kappa L}$ term to match in $x > L$



for $E = V_0$, now negative slope is achieved, can match to <u>dying</u> exponential $(Be^{-\kappa x})$

$V_0 > \frac{\hbar^2 \pi^2}{8mL^2}$

a famous physical example:

(proton + neutron) = successful bound state, but just barely.

(neutron + neutron) = just fails as a bound state; potential too shallow and/or narrow.

Probability Current

→ stay in one dimension (book does 3).

Schrödinger: $$i\hbar \frac{d\psi}{dt} = -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \underset{\uparrow\ \text{real}}{V(x)}\,\psi \quad \text{(A)}$$

Complex Conjugate: $$-i\hbar \frac{d\psi^*}{dt} = \frac{-\hbar^2}{2m}\frac{d^2\psi^*}{dx^2} + V(x)\,\psi^* \quad \text{(B)}$$

multiply (A) by ψ^* and (B) by ψ, then subtract (B) from (A):

$$\underbrace{i\hbar\left(\psi^*\frac{d\psi}{dt} + \psi\frac{d\psi^*}{dt}\right)} = \frac{-\hbar^2}{2m}\left(\psi^*\frac{d^2\psi}{dx^2} - \psi\frac{d^2\psi^*}{dx^2}\right)$$

$$i\hbar\frac{d}{dt}|\psi(x)|^2 = \frac{-\hbar^2}{2m}\frac{d}{dx}\left(\psi^*\frac{d\psi}{dx} - \psi\frac{d\psi^*}{dx}\right)$$

and now integrate over some region, x from a to b:

$$\frac{d}{dt}\underbrace{\int_a^b dx\,|\psi(x)|^2}_{\text{probability between } a + b} = \frac{i\hbar}{2m}\int_a^b dx\,\frac{d}{dx}\left(\psi^*\frac{d\psi}{dx} - \psi\frac{d\psi^*}{dx}\right)$$

$$= \underbrace{\frac{i\hbar}{2m}\left(\psi^*\frac{d\psi}{dx} - \psi\frac{d\psi^*}{dx}\Bigg|_a^b\right)}_{\text{net current; in at point } b\text{, out at point } a}$$

time derivative

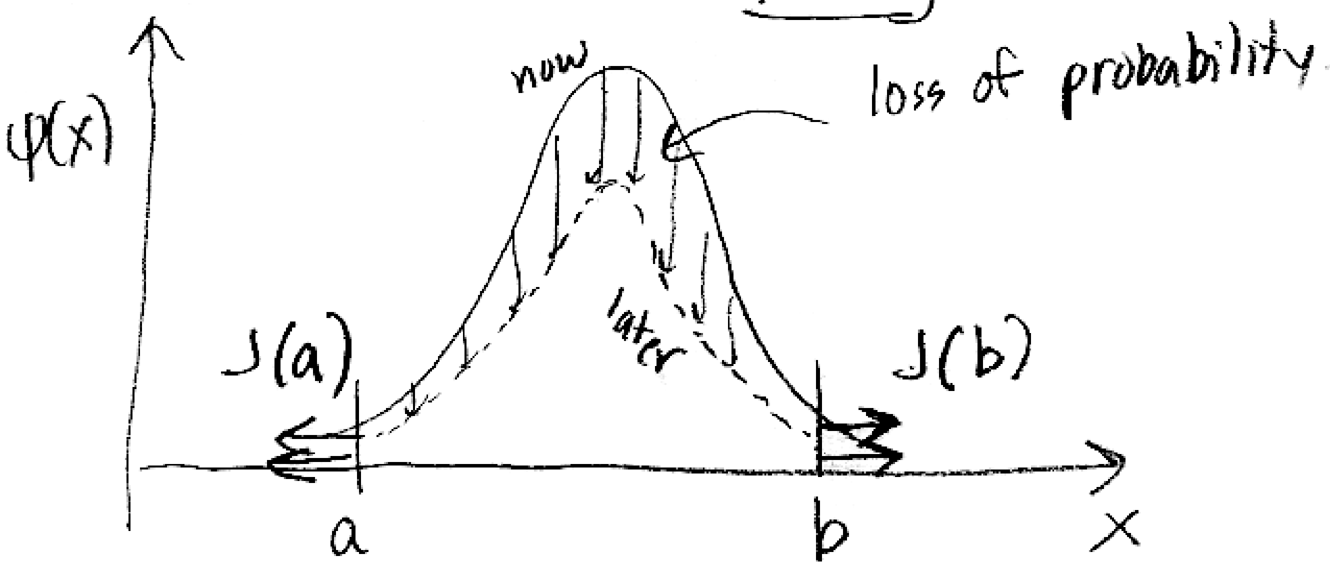
$$J \equiv -\frac{i\hbar}{2m}\left(\psi^* \frac{d\psi}{dx} - \psi \frac{d\psi^*}{dx}\right)\Big|_a^b$$

defined as "probability current"

so

$$\frac{d}{dt}\int_a^b dx\,|\psi(x)|^2 = J(a) - \underline{J(b)}$$

when $J(b) > 0$, current leaving



The Classical Limit

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Chap. #6

look at the time derivative of the expectation value of an operator $\underset{\sim}{\Omega}$:

$$\frac{d}{dt}\langle\psi|\underset{\sim}{\Omega}|\psi\rangle = \frac{d\langle\psi|}{dt}\underset{\sim}{\Omega}|\psi\rangle + \langle\psi|\frac{d\underset{\sim}{\Omega}}{dt}|\psi\rangle + \langle\psi|\underset{\sim}{\Omega}\frac{d|\psi\rangle}{dt}$$

But the time derivatives of the bras & kets satisfy the Schrödinger Equation:

$$\frac{d|\psi\rangle}{dt} = \frac{\underset{\sim}{H}}{i\hbar}|\psi\rangle \rightarrow \frac{d\langle\psi|}{dt} = -\langle\psi|\frac{\underset{\sim}{H}^{+}}{i\hbar}$$

$-$ sign is from $i\hbar$

but $\underset{\sim}{H}^{+} = \underset{\sim}{H}$ (Hamiltonian is Hermitian)

so $$\frac{d\langle\psi|}{dt} = -\frac{1}{i\hbar}\langle\psi|\underset{\sim}{H}$$

so,

$$\frac{d}{dt}\langle\underset{\sim}{\Omega}\rangle = -\frac{1}{i\hbar}\langle\psi|\underset{\sim}{H}\underset{\sim}{\Omega}|\psi\rangle + \langle\psi|\frac{d\underset{\sim}{\Omega}}{dt}|\psi\rangle + \frac{1}{i\hbar}\langle\psi|\underset{\sim}{\Omega}\underset{\sim}{H}|\psi\rangle$$

$$= \frac{1}{i\hbar}\langle\psi|(\underset{\sim}{\Omega}\underset{\sim}{H} - \underset{\sim}{H}\underset{\sim}{\Omega})|\psi\rangle + \langle\psi|\frac{d\underset{\sim}{\Omega}}{dt}|\psi\rangle$$

$$\boxed{\frac{d}{dt}\langle\underset{\sim}{\Omega}\rangle = \frac{1}{i\hbar}\langle[\underset{\sim}{\Omega},\underset{\sim}{H}]\rangle + \langle\psi|\frac{d\underset{\sim}{\Omega}}{dt}|\psi\rangle}$$

"Ehrenfest's Theorem"

Examples (all one-dimensional)

$\boxed{\underset{\sim}{\Omega} = \underset{\sim}{x}}$ and then $\frac{d\underset{\sim}{\Omega}}{dt} = 0$

$$\underset{\sim}{H} = \frac{p^2}{2m} + V(\underset{\sim}{x})$$

$$\frac{d}{dt}\langle \underset{\sim}{x} \rangle = \frac{1}{i\hbar}\langle [\underset{\sim}{x}, \underset{\sim}{H}] \rangle \qquad [\underset{\sim}{x}, \underset{\sim}{H}] = [\underset{\sim}{x}, (\frac{p^2}{2m} + V(\underset{\sim}{x}))]$$

$$= [\underset{\sim}{x}, \frac{p^2}{2m}] + \underbrace{\cancel{[\underset{\sim}{x}, V(\underset{\sim}{x})]}}_{0}$$

$[\underset{\sim}{x}, p^2]$ ← look at p. 20, 1.5.10

$$= p\underbrace{[\underset{\sim}{x}, p]}_{i\hbar} + \underbrace{[\underset{\sim}{x}, p]}_{i\hbar}p = 2i\hbar p$$

so $$\frac{d}{dt}\langle \underset{\sim}{x} \rangle = \frac{1}{i\hbar}\langle [\underset{\sim}{x}, \underset{\sim}{H}] \rangle = \frac{1}{i\hbar}\frac{1}{2m}\langle [\underset{\sim}{x}, p^2] \rangle$$

$$= \frac{1}{\cancel{i\hbar}}\frac{1}{\cancel{2}m} \cdot \cancel{2i\hbar}\langle p \rangle$$

$$\frac{d\langle \underset{\sim}{x} \rangle}{dt} = \frac{\langle p \rangle}{m} \qquad \left(\frac{dx}{dt} = \frac{p}{m} \text{ is classical equation}\right)$$

⇒ what holds for classical variables holds for EXPECTATION VALUES in QM.

$\boxed{\underset{\sim}{\Omega} = p}$ ⇒ expect, $\frac{dp}{dt} = -\frac{dV}{dx}$ (classical)

so, $\frac{d\langle p \rangle}{dt} = -\left\langle \frac{dV}{dx} \right\rangle$

$$\frac{d}{dt}\langle p \rangle = \frac{1}{i\hbar}\langle [p, \underset{\sim}{H}] \rangle = \frac{1}{i\hbar}\left(\underbrace{\langle [p, \frac{p^2}{2m}] \rangle}_{0} + \langle [p, V(\underset{\sim}{x})] \rangle\right)$$

$$[P, V(X)]|\psi\rangle \doteq \frac{\hbar}{i}\frac{d}{dx}V(x)\psi(x) - V(x)\frac{\hbar}{i}\frac{d}{dx}\psi(x)$$

$$\doteq \frac{\hbar}{i}\frac{dV}{dx}\psi + \cancel{\frac{\hbar}{i}V\frac{d\psi}{dx}} - \cancel{V\frac{\hbar}{i}\frac{d\psi}{dx}}$$

$$= \frac{\hbar}{i}\frac{dV}{dX}|\psi\rangle$$

so, $\boxed{\frac{d}{dt}\langle P\rangle = \frac{1}{i\hbar}\frac{\hbar}{i}\left\langle\frac{dV}{dX}\right\rangle = -\left\langle\frac{dV}{dx}\right\rangle}$

as expected

$\boxed{\Omega = H}$ suppose H = independent of time

$$\frac{d}{dt}\langle H\rangle = \frac{1}{i\hbar}\langle[H, H]\rangle = 0$$

<u>mean</u> energy of system is conserved.

However, it appears sometimes that energy is not conserved in specific cases. Usually this kind of apparent situation results from interaction of measurement apparatus with the system, or, broadly, the focusing on a subsystem instead of the entire system.