Chapter 4 (p. 115)

Postulates of Non-relativistic QM

Classical M.	Quantum M.
(I) Describe a point particle:	
$x(t)$, $p(t)$ } note, you may be more familiar with $x, \dot{x}$	$\vert\Psi(t)\rangle$
Lagrangian → use $x, \dot{x}$	
Hamiltonian → use x, p	
An "extended" particle or system: $\vec{\theta}, \vec{\omega}$ (Lagrangian); $\vec{\theta}, \vec{L}$ (Hamiltonian)	add dimensions to $\vert\Psi(t)\rangle$

(II) "Dynamical Variables," like, kinetic energy, are a function of x, p

Kinetic Energy $T = \frac{p^2}{2m}$

Generally $\omega = \omega(x, p)$

"Intrinsic" angular momentum
→ 3-dimensional
→ "spin"
→ S_x, S_y, S_z

Dynamical Variables are "represented" by Hermitian Operators

one dimension $\begin{cases} x \to \underset{\sim}{X}, \ \langle x|\underset{\sim}{X}|x'\rangle = x\delta(x-x') \\ p \to \underset{\sim}{P} = \hbar \underset{\sim}{K} \ \langle x|\underset{\sim}{P}|x'\rangle = \frac{\hbar}{i}\delta'(x-x') \end{cases}$

$\underset{\sim}{\Omega}(\underset{\sim}{X}, \underset{\sim}{P}) = \omega(x \to \underset{\sim}{X}, p \to \underset{\sim}{P})$

3-d, but QM easier

$\underset{\sim}{S}_z \doteq \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad \underset{\sim}{S}_y \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\underset{\sim}{S}_x \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$|\vec{\underset{\sim}{S}}|^2 = \underset{\sim}{S}_x^2 + \underset{\sim}{S}_y^2 + \underset{\sim}{S}_z^2 = \frac{\hbar^2}{4} \cdot 3 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Classical	Quantum
(III) Measurements of $x, p, \omega(x,p)$ don't modify state of particle or system.	Measurement of $\underset{\sim}{\Omega}$ yields only an eigenvalue of $\underset{\sim}{\Omega}$ ... call that ω. The probability of seeing ω is $P(\omega) \propto \lvert\langle\omega\vert\psi\rangle\rvert^2$, where $\vert\psi\rangle$ is the state of the system prior to measurement. After measurement, system is in state $\vert\omega\rangle$.

The biggest one

(IV) Equation of motion:

$$\dot{x} = \frac{\partial H}{\partial p} \quad \dot{p} = -\frac{\partial H}{\partial x}$$

may be unfamiliar.

$$L = T - V \text{ (recall)}$$

$$H = T + V$$

$$= \underbrace{\frac{1}{2} m\dot{x}^2}_{p = m\dot{x}} + V(x)$$

$$H = \frac{1}{2}\frac{p^2}{m} + V$$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}.$$

$$\dot{p} = \frac{-\partial H}{\partial x} = -\frac{\partial V}{\partial x}$$

(Newton).

Equation of motion

$$i\hbar \frac{d}{dt}\vert\psi(t)\rangle = \underset{\sim}{H}\vert\psi(t)\rangle$$

for example, for non-relativistic 1-d:

$$\underset{\sim}{H} = \frac{p^2}{2m} + V(\underset{\sim}{x})$$

I'm a little dissatisfied with Shankar's discussion, pp. 117 - 127. Instead, I'll focus on a 2-dimensional <u>example</u>. "SPIN"

$$\underset{\sim}{S}_z \doteq \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad \underset{\sim}{S}_y \doteq \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \qquad \underset{\sim}{S}_x \doteq \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$|\psi\rangle \doteq \begin{pmatrix}a\\ \sqrt{1-a^2}\end{pmatrix} \qquad a \to \text{real } \#, \leq 1$$

measure $\underset{\sim}{S}_z$: 1) what are eigenvalues of $\underset{\sim}{S}_z$?

$$+\frac{\hbar}{2}, \quad |+\tfrac{\hbar}{2}\rangle \doteq \begin{pmatrix}1\\0\end{pmatrix}$$

$$-\frac{\hbar}{2}, \quad |-\tfrac{\hbar}{2}\rangle \doteq \begin{pmatrix}0\\1\end{pmatrix}$$

2) initial state $\doteq \begin{pmatrix}a\\ \sqrt{1-a^2}\end{pmatrix}$, now "measure" $\underset{\sim}{S}_z$.

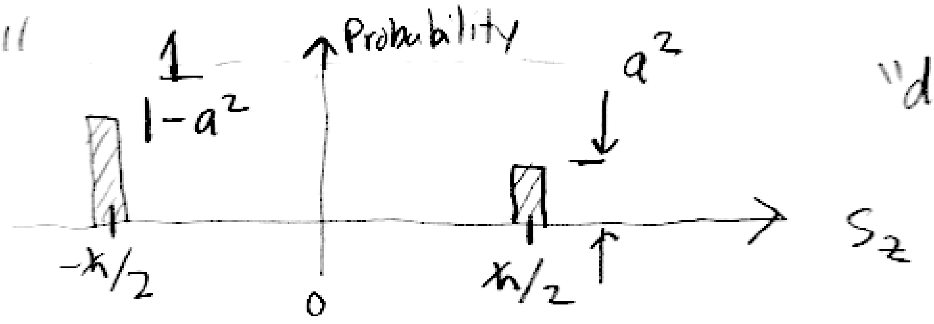
(a) get $+\frac{\hbar}{2}$ (spin-up) with probability:

$$|\langle+\tfrac{\hbar}{2}|\psi\rangle|^2 = \left|(1 \;\; 0)\begin{pmatrix}a\\ \sqrt{1-a^2}\end{pmatrix}\right|^2 = a^2 \Rightarrow \text{now state is } \begin{pmatrix}1\\0\end{pmatrix}!$$

(b) get $-\frac{\hbar}{2}$ (spin-down) with probability

$$|\langle-\tfrac{\hbar}{2}|\psi\rangle|^2 = \left|(0 \;\; 1)\begin{pmatrix}a\\ \sqrt{1-a^2}\end{pmatrix}\right|^2 = 1-a^2 \Rightarrow \text{new state is } \begin{pmatrix}0\\1\end{pmatrix}!$$

<u>never</u> measure $0, \frac{2}{3}\cdot\frac{\hbar}{2}, \frac{1}{2}\cdot\frac{\hbar}{2}, -\frac{4}{5}\frac{\hbar}{2}$

"Histogram"  "digital"

This is weird! What can have an "analog" value is the average value of S_z. This is called the expectation value of $\underset{\sim}{S}_z$.

Here: expectation: $\underbrace{\langle \psi | \underset{\sim}{S}_z | \psi \rangle}_{\text{Just notation now}} = \frac{\hbar}{2} \cdot (a)^2 + \left(-\frac{\hbar}{2}\right)\left(1-(a)^2\right)$

value, probability, value, probability

$$\text{"}\langle \psi | \underset{\sim}{S}_z | \psi \rangle\text{"} = \frac{\hbar}{2} \cdot \left((a)^2 + (a)^2\right) - \frac{\hbar}{2} = \frac{\hbar}{2}\left(2a^2 - 1\right)$$

note: $a = \pm 1$, average $= +\hbar/2$ (never see $-\hbar/2$)

$a = 0$ average $= -\hbar/2$ (never see $+\hbar/2$)

another way to view "Expectation Value":

$$a^2 = \left|\langle +\tfrac{\hbar}{2} | \psi \rangle\right|^2 \qquad 1 - a^2 = \left|\langle -\tfrac{\hbar}{2} | \psi \rangle\right|^2$$

so

$$\frac{\hbar}{2} \cdot a^2 = \frac{\hbar}{2} \langle \psi | +\tfrac{\hbar}{2} \rangle \langle +\tfrac{\hbar}{2} | \psi \rangle \qquad \left(-\frac{\hbar}{2}\right)(1-a^2) = -\frac{\hbar}{2} \langle \psi | -\tfrac{\hbar}{2} \rangle \langle -\tfrac{\hbar}{2} | \psi \rangle$$

$$\underset{\sim}{S}_z | +\tfrac{\hbar}{2} \rangle = \frac{\hbar}{2} | +\tfrac{\hbar}{2} \rangle \qquad \underset{\sim}{S}_z | -\tfrac{\hbar}{2} \rangle = -\frac{\hbar}{2} | -\tfrac{\hbar}{2} \rangle$$

$$\langle \tfrac{\hbar}{2} | \underset{\sim}{S}_z | \tfrac{\hbar}{2} \rangle = \frac{\hbar}{2} \langle +\tfrac{\hbar}{2} | +\tfrac{\hbar}{2} \rangle = \frac{\hbar}{2} \qquad \langle -\tfrac{\hbar}{2} | \underset{\sim}{S}_z | -\tfrac{\hbar}{2} \rangle = -\frac{\hbar}{2} \langle -\tfrac{\hbar}{2} | -\tfrac{\hbar}{2} \rangle = -\frac{\hbar}{2}$$

so $\frac{\hbar}{2} a^2 = \langle \psi | +\tfrac{\hbar}{2} \rangle \langle +\tfrac{\hbar}{2} | \underset{\sim}{S}_z | +\tfrac{\hbar}{2} \rangle \langle +\tfrac{\hbar}{2} | \psi \rangle$

$$\left(-\frac{\hbar}{2}\right)(1-a^2) = \langle \psi | -\tfrac{\hbar}{2} \rangle \langle -\tfrac{\hbar}{2} | \underset{\sim}{S}_z | -\tfrac{\hbar}{2} \rangle \langle \psi | -\tfrac{\hbar}{2}$$

Average Value = Expectation Value

$$= \frac{\hbar}{2} a^2 + \left(-\frac{\hbar}{2}\right)(1-a^2)$$

$$= \langle \psi | +\tfrac{\hbar}{2} \rangle \langle +\tfrac{\hbar}{2} | \underset{\sim}{S_z} | +\tfrac{\hbar}{2} \rangle \langle +\tfrac{\hbar}{2} | \psi \rangle$$

$$+ \langle \psi | -\tfrac{\hbar}{2} \rangle \langle -\tfrac{\hbar}{2} | \underset{\sim}{S_z} | -\tfrac{\hbar}{2} \rangle \langle -\tfrac{\hbar}{2} | \psi \rangle$$

but: $|\tfrac{\hbar}{2}\rangle\langle\tfrac{\hbar}{2}| + |-\tfrac{\hbar}{2}\rangle\langle-\tfrac{\hbar}{2}| = \underset{\sim}{\mathbb{1}}$

also: $\langle +\tfrac{\hbar}{2} | \underset{\sim}{S_z} | -\tfrac{\hbar}{2} \rangle = \langle +\tfrac{\hbar}{2} | \left(-\tfrac{\hbar}{2}\right) | -\tfrac{\hbar}{2} \rangle = -\tfrac{\hbar}{2} \langle +\tfrac{\hbar}{2} | -\tfrac{\hbar}{2} \rangle = 0$

$\langle -\tfrac{\hbar}{2} | \underset{\sim}{S_z} | +\tfrac{\hbar}{2} \rangle = \langle -\tfrac{\hbar}{2} | \left(\tfrac{\hbar}{2}\right) | +\tfrac{\hbar}{2} \rangle = +\tfrac{\hbar}{2} \langle -\tfrac{\hbar}{2} | +\tfrac{\hbar}{2} \rangle = 0$

so

Expectation Value $= \langle \psi | \underset{\sim}{\mathbb{1}} \, \underset{\sim}{S_z} \, \underset{\sim}{\mathbb{1}} | \psi \rangle = \langle \psi | \underset{\sim}{S_z} | \psi \rangle$.

<u>Another way</u>: $\langle \underset{\sim}{\Omega} \rangle = \sum_{i=1}^{n} P(\omega_i)\, \omega_i$

brackets mean average

probability of ω_i

$$\sum_{i=1}^{n} P(\omega_i) = \underline{1}$$

but in state $|\psi\rangle$, $P(\omega_i) = |\langle \psi | \omega_i \rangle|^2$

$$= \langle \psi | \omega_i \rangle \langle \omega_i | \psi \rangle$$

so $\langle \underset{\sim}{\Omega} \rangle = \sum_{i=1}^{n} \langle \psi | \omega_i \rangle \langle \omega_i | \psi \rangle \omega_i$

$$= \sum_{i=1}^{n} \langle \psi | \underbrace{\omega_i | \omega_i \rangle}_{= \underset{\sim}{\Omega} | \omega_i \rangle} \langle \omega_i | \psi \rangle = \sum_{i=1}^{n} \langle \psi | \underset{\sim}{\Omega} \underbrace{| \omega_i \rangle \langle \omega_i |}_{\underset{\sim}{\mathbb{1}}} \psi \rangle$$

$$= \langle \psi | \underset{\sim}{\Omega} | \psi \rangle$$

Important Concept:

The Classical notion of S_z (or of any dynamical variable) corresponds in Quantum to the EXPECTATION VALUE of $\underset{\sim}{S}_z$, $\langle\psi|\underset{\sim}{S}_z|\psi\rangle$

expectation value: anything $-\frac{\hbar}{2}$ to $+\frac{\hbar}{2}$

individual measurements: only $-\frac{\hbar}{2}$ or $\frac{\hbar}{2}$

Quantum Concept "The Uncertainty" $\equiv$ Variance

$$(\Delta\underset{\sim}{\Omega})^2 \equiv \sum_{i=1}^{n} (\omega_i - \langle\underset{\sim}{\Omega}\rangle)^2 P(\omega_i)$$

1) if $P(\omega_j) = 1 \quad i=j$ | $\langle\underset{\sim}{\Omega}\rangle = \omega_j$
$P(\omega_i) = 0 \quad i\neq j$ | $(\Delta\underset{\sim}{\Omega})^2 = (\omega_j - \omega_j)^2 \times 1 = 0$

2) $(\Delta\underset{\sim}{\Omega})^2 \geq 0$

3) $(\Delta\underset{\sim}{\Omega})^2 > 0$ when measurements yield at least two distinct values

Example: $|\psi\rangle \doteq \begin{pmatrix} a \\ \sqrt{1-a^2} \end{pmatrix}$ $\quad \langle\underset{\sim}{S}_z\rangle = \left(\frac{\hbar}{2}\right)(2a^2-1)$

$$(\Delta \underset{\sim}{S}_z)^2 = \Big(\underbrace{\tfrac{\hbar}{2}}_{\text{eigenvalue}} - \underbrace{\left(\tfrac{\hbar}{2}\right)(2a^2-1)}_{\langle \underset{\sim}{S}_z\rangle}\Big)^2 \underbrace{a^2}_{\text{prob of } +\hbar/2} + \Big(\underbrace{-\tfrac{\hbar}{2}}_{\text{eigenvalue}} - \underbrace{\left(\tfrac{\hbar}{2}\right)(2a^2-1)}_{\langle \underset{\sim}{S}_z\rangle}\Big)^2 \underbrace{(1-a^2)}_{\text{prob of } -\hbar/2}$$

$$= \left(\frac{\hbar}{2}\right)^2 \Big\{ (\underbrace{1-2a^2+1}_{2(1-a^2)})^2 a^2 + (\underbrace{-1-2a^2+1}_{-2a^2})^2 (1-a^2) \Big\}.$$

$$= \left(\frac{\hbar}{2}\right)^2 \left\{ 4(1-a^2)^2 a^2 + 4a^4(1-a^2) \right\} = \left(\frac{\hbar}{2}\right)^2 \cdot 4a^2(1-a^2)(1-a^2+a^2)$$

$$(\Delta \underset{\sim}{S}_z)^2 = \left(\frac{\hbar}{2}\right)^2 4a^2(1-a^2) \qquad a \text{ real}, \leqslant 1$$

$$\boxed{\Delta \underset{\sim}{S}_z \equiv \sqrt{(\Delta \underset{\sim}{S}_z)^2} = \left(\frac{\hbar}{2}\right) \cdot 2a\sqrt{1-a^2}}$$

$= 0$ when $a = 1$ or $a = 0$

More interesting: measure $\underset{\sim}{S}_x$!

page 117

(2) Find eigenvalues, eigenvectors:

$$\underset{\sim}{S}_x \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow \begin{vmatrix} -\omega & \hbar/2 \\ \hbar/2 & -\omega \end{vmatrix} = \omega^2 - \left(\frac{\hbar}{2}\right)^2 = 0$$

$$\omega = \pm \hbar/2$$

$+\hbar/2$: $\begin{pmatrix} -\hbar/2 & +\hbar/2 \\ +\hbar/2 & -\hbar/2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$ so, $\alpha = \beta$

eigenvector: $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$-\hbar/2$ $\begin{pmatrix} \hbar/2 & \hbar/2 \\ \hbar/2 & \hbar/2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$ so, $\alpha = -\beta$

eigenvector $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(3) say $|\psi\rangle \doteq \begin{pmatrix} a \\ \sqrt{1-a^2} \end{pmatrix}$

$$= \sum_{i=1}^{2} |\omega_{xi}\rangle \underbrace{\langle \omega_{xi} | \psi \rangle}$$

note: $\langle +\frac{\hbar}{2}x | \psi \rangle = \frac{1}{\sqrt{2}}(1 \ 1)\begin{pmatrix} a \\ \sqrt{1-a^2} \end{pmatrix} = \frac{1}{\sqrt{2}}(a + \sqrt{1-a^2})$

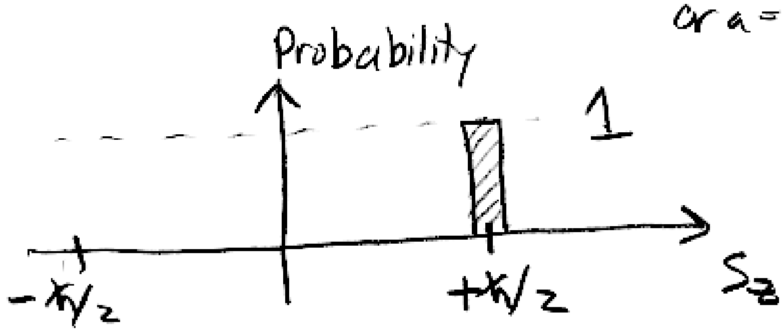
$\langle -\frac{\hbar}{2}x | \psi \rangle = \frac{1}{\sqrt{2}}(1 \ -1)\begin{pmatrix} a \\ \sqrt{1-a^2} \end{pmatrix} = \frac{1}{\sqrt{2}}(a - \sqrt{1-a^2})$

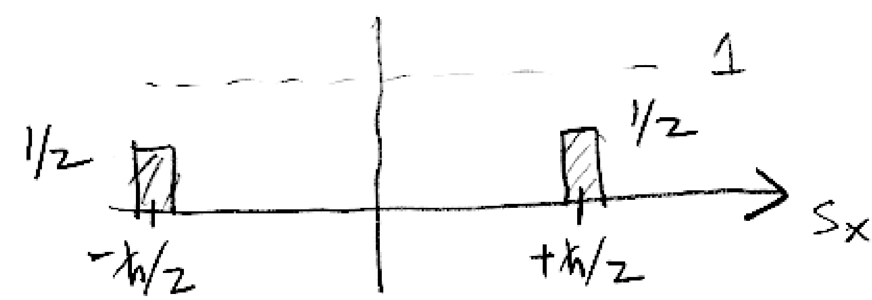
(4) Probability that S_x of $+\hbar/2$ measured:

$|\langle +\frac{\hbar}{2}x | \psi \rangle|^2 = \frac{1}{2}(a + \sqrt{1-a^2})^2$, $= \frac{1}{2}$ when $a=1$ or $a=0$

$|\langle -\frac{\hbar}{2}x | \psi \rangle|^2 = \frac{1}{2}(a - \sqrt{1-a^2})^2$, $= \frac{1}{2}$ when $a=1$ or $a=0$

Suppose $a=1$ ....





S_z is "sharp"

S_x is "blurry".