Infinite Dimensions

p.57 1.10

can describe a continuous function $f(x)$ by its evaluations at a set of points:

$$\begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix} \doteq |f_n\rangle$$

note: $\Delta = x_{i+1} - x_i \simeq \frac{L}{n}$

both: $L, n \to \infty$ possible.

can visualize:

$$\begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix} = f(x_1)\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + f(x_2)\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \dots + f(x_n)\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$|f_n\rangle = f(x_1)\,|x_1\rangle + f(x_2)\,|x_2\rangle + \dots + f(x_n)\,|x_n\rangle$$

orthonormal $\langle x_i | x_j \rangle = \delta_{ij}$

complete (in interval) $\sum_i |x_i\rangle\langle x_i| = \underset{\sim}{\mathbb{1}}$

note then

$$|f_n\rangle = \underset{\sim}{\mathbb{1}} = \sum_i |x_i\rangle\langle x_i|f_n\rangle \quad \text{an } f(x_i) = \langle x_i|f_n\rangle$$

"Measure" and "Normalization", infinitesimals.

So far... norm of $\{|x_i\rangle\}$ looks good

$|f_n\rangle$? TRY

$$\langle f_n|f_n\rangle = [f^*(x_1)\, f^*(x_2) \ldots f^*(x_n)] \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix}$$

$$= \sum_{i=1}^{n} |f_n(x_i)|^2$$

Think if $f_n(x_i) =$ constant $\equiv \alpha$

then $\langle f_n|f_n\rangle = |\alpha|^2 \sum_{i=1}^{n} 1 = n|\alpha|^2$

now imagine letting $n \to \infty$ ... then with this "TRY" for $\langle f_n|f_n\rangle$, $\langle f_n|f_n\rangle \to \infty$

$\boxed{\text{BAD TRY}}$.

TRY AGAIN : $\langle f_n|f_n\rangle = \overset{\Delta =}{\left(\frac{L}{n}\right)} (f^*(x_1) f^*(x_2) \ldots f^*(x_n)) \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix}$

then, when $f_n(x_i) = \alpha$

$$\langle f_n|f_n\rangle = \frac{L}{\cancel{n}} \times \cancel{n} \times |\alpha|^2$$

$\boxed{\langle f_n|f_n\rangle = |\alpha|^2 \cdot L}$ nice

Recognize then...

$$\lim_{n\to\infty} \langle f_n | f_n \rangle = \lim_{n\to\infty} \left(\frac{L}{n}\right) \underbrace{[f^*(x_1) f^*(x_2) \ldots f^*(x_n)]}_{\leftarrow \text{ gets bigger as } n\to\infty \rightarrow} \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix}$$

$$= \int_0^L dx |f(x)|^2$$

and (p. 59)

$$\langle f | g \rangle = \int_a^b dx\, f^*(x)\, g(x) \qquad \text{p. 59} \; 1.10.9$$

what now happens to $\langle x_i | x_j \rangle$ and $\sum_{i=1}^{n} |x_i\rangle\langle x_i|$?

look above, note: $g(x) = \langle x | g \rangle$ $f^*(x) = \langle f | x \rangle$

$$\langle f | g \rangle = \int_a^b dx \langle f | x \rangle \langle x | g \rangle$$

$$\underset{\sim}{1} = \int_a^b dx\, |x\rangle\langle x| \qquad \left(\underline{b, a \text{ could}} \text{ be } \pm\infty\right)$$

(1.10.11 p. 59).

This is a renormalization of the $|x_i\rangle$ ket that applies as $n \to \infty$...

To see this, consider inserting unity on the relationship:

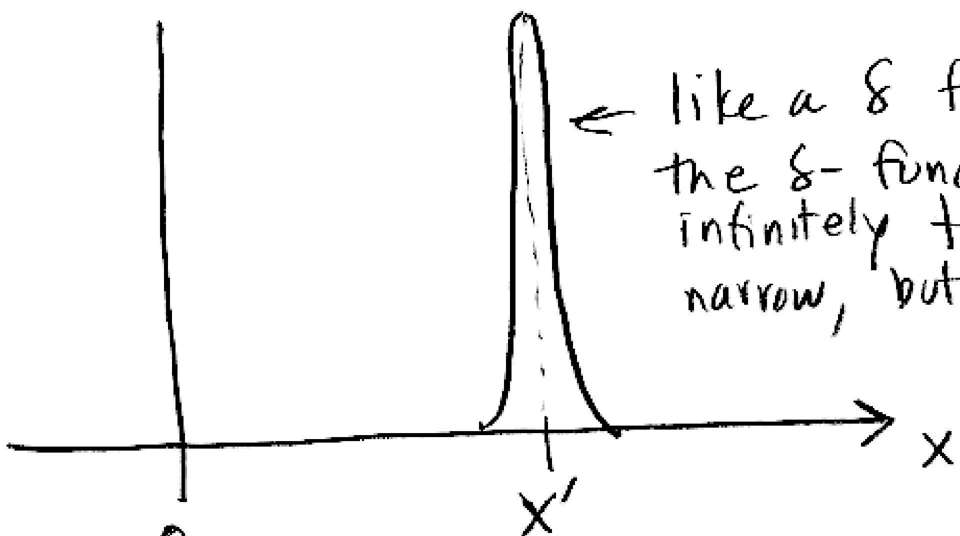
$$\langle x | f \rangle = f(x)$$

$$\underset{\sim}{\mathbb{1}} = \int dx' \langle x | x' \rangle \underbrace{\langle x' | f \rangle}_{f(x')} = f(x)$$

so $\int dx' \langle x | x' \rangle f(x') = f(x)$

for just about any $f(x)$.... you recognise:

thought of as a function of x → $\langle x | x' \rangle = \delta(x - x')$ "Delta Function"



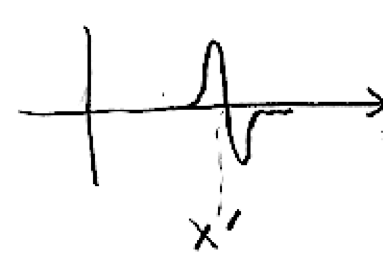
← like a δ function, but the δ- function is infinitely tall, infinitely narrow, but has area of 1 unit

from above: $\int dx' \delta(x - x') 1 = 1$

need only integrate between $x' = x - \varepsilon$ and $x' = x + \varepsilon$ with ε very, very small

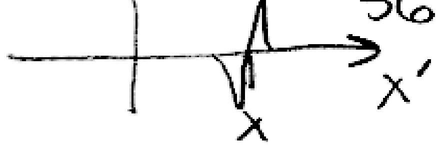
Conceptually, can differentiate $\delta(x - x')$:

$$\delta'(x - x') \equiv \frac{d}{dx} \delta(x - x')$$



LOOK OUT! $\neq \frac{d}{dx'} \delta(x - x')$

but $\frac{d}{dx'}\delta(x-x') = -\delta'(x-x')$



and $\int dx'\, \delta'(x-x') f(x') = f'(x)$

nth derivative: $\int dx'\, \delta^{(n)}(x-x') f(x') = \frac{d^n f(x)}{dx^n}$

Fourier Tranforms as a Basis Change

Recall: $\tilde{f}(k) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-ikx} f(x)$

my notation

wave number

now $b = \infty$
$a = -\infty$

View this as a different "representation" of f.

$\tilde{f}(k) = \langle k|f\rangle = \int_{-\infty}^{\infty} dx \langle k|x\rangle \underbrace{\langle x|f\rangle}_{f(x)}$

$\hookrightarrow$ insert $\underset{\sim}{\mathbb{1}}$

$\hookrightarrow \langle k|x\rangle \equiv \frac{1}{\sqrt{2\pi}} e^{-ikx}$

now more fun:

$\langle x|f\rangle = \int_{-\infty}^{\infty} dk \underbrace{\langle x|k\rangle}_{=\langle k|x\rangle^* = \frac{1}{\sqrt{2\pi}} e^{ikx}} \overbrace{\langle k|f\rangle}^{\tilde{f}(k)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{ikx} \tilde{f}(k)$

$\uparrow$

$\underset{\sim}{\mathbb{1}} = \int dk |k\rangle\langle k|$

inversion.

more fun:

$$\langle x | x' \rangle = \int_{-\infty}^{\infty} dk \, \langle x | k \rangle \langle k | x' \rangle$$

$$\frac{1}{\sqrt{2\pi}} e^{+ikx} \qquad \frac{1}{\sqrt{2\pi}} e^{-ikx'}$$

and so: $$\delta(x-x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, e^{ik(x-x')}$$

$$\delta(k-k') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \, e^{ix(k-k')}$$

$$= \frac{1}{2\pi} \int dx \, \langle k' | x \rangle \langle x | k \rangle$$

$$\delta(k-k') = \langle k | k' \rangle$$

Operators for Continuous/Infinite Case

$|f\rangle \doteq \langle x|f\rangle = f(x)$ basis is $|x\rangle$
x continuous

what operator is the x basis an eigenbasis for? $\underset{\sim}{x}$

so $\underset{\sim}{x}|x'\rangle = x'|x'\rangle$

$\uparrow$ eigenvalue ..

visual...

in terms of functions (representations)

$$\underbrace{\langle x|\underset{\sim}{x}|x'\rangle}_{\text{function that represents } \underset{\sim}{x}|x'\rangle} = \underbrace{\langle x|x'|x'\rangle}_{\text{since } |x'\rangle \text{ is eigenvalue of } \underset{\sim}{x}} = x'\underbrace{\langle x|x'\rangle}_{\text{function that represents } |x'\rangle} = x'\delta(x-x')$$

x' a parameter
x a variable

note ... $x'\delta(x-x') = x\delta(x-x')$

why? consider $\int dx\, x'\delta(x-x')f(x)$

$$= x'\int dx\,\delta(x-x')f(x)$$
$$= x'f(x')$$

also $\int dx\, x\delta(x-x')f(x) = \int dx\,\delta(x-x')\big(xf(x)\big)$

$$= x'f(x')$$

conclude $\langle x|\underset{\sim}{x}|x'\rangle = x'\delta(x-x') = x\,\delta(x-x')$

meaning rep of $\langle x|\underset{\sim}{x}$ is... $x\,\delta(x-x')$ ... so,

$$\langle x|\underset{\sim}{x}|x'\rangle = x\langle x|x'\rangle$$

or $\langle x|\underset{\sim}{x} = x\langle x|$

- $\langle x|$ is eigenbra of $\underset{\sim}{x}$.
- $\underset{\sim}{x}$ is hermitian.

How abut....

$$\underset{\sim}{x}|f\rangle \doteq \langle x|\underset{\sim}{x}|f\rangle$$

$$= \int dx' \langle x|\underset{\sim}{x}|x'\rangle\langle x'|f\rangle$$

$$= \int dx'\, x'\,\delta(x-x')\,f(x')$$

$$\underset{\sim}{x}|f\rangle \doteq x f(x)$$

sometimes say... $\underset{\sim}{x}|f\rangle = |xf\rangle$

and $\underset{\sim}{x} f(x) = x f(x)$