Some Degeneracy Examples

$$\begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} \xrightarrow{\text{diagonalize}} \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} = \omega \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

diagonal in <u>any</u> orthonormal basis

($\Omega_{12} = \Omega_{21} = 0$ in this case).

$$\begin{pmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} \end{pmatrix} \xrightarrow{\text{diagonalize}} \begin{pmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_1 & 0 \\ 0 & 0 & \omega_2 \end{pmatrix} = \omega_1 \begin{array}{c} \text{①} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array} + \omega_2 \begin{array}{c} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \text{②} \end{array}$$

$\omega_1 \neq \omega_2$

- in 2×2 subspace ①, <u>any</u> basis maintains diagonality.
- must use specific eigenvector for ②.
- subspace of ① $\perp$ to subspace of ②

$$\begin{pmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} \end{pmatrix} \xrightarrow{\text{diag}} \begin{pmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_1 \end{pmatrix}$$

$\omega_1 \neq \omega_2$

← actually, same as above.

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{U^\dagger} \begin{pmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \omega_1 & 0 & 0 \\ 0 & 0 & \omega_2 \\ 0 & \omega_1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_1 & 0 \\ 0 & 0 & \omega_2 \end{pmatrix}$$

$$\begin{pmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} & \Omega_{24} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} & \Omega_{34} \\ \Omega_{41} & \Omega_{42} & \Omega_{43} & \Omega_{44} \end{pmatrix} \xrightarrow{\text{diag}} \begin{pmatrix} w_1 & 0 & 0 & 0 \\ 0 & w_2 & 0 & 0 \\ 0 & 0 & w_1 & 0 \\ 0 & 0 & 0 & w_2 \end{pmatrix} \quad w_1 \neq w_2$$

use additional $U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$\rightarrow \begin{pmatrix} w_1 & 0 & 0 & 0 \\ 0 & w_1 & 0 & 0 \\ 0 & 0 & w_2 & 0 \\ 0 & 0 & 0 & w_2 \end{pmatrix}$$

- 2 $\perp$ 2x2 subspaces
- within each subspace, any ortnonormal basis works.

Degeneracy, inside the denerate subspace, is easier to handle than diagonalization!

Unitary Diagonalization

$$\underset{\sim}{U}|u_i\rangle = u_i|u_i\rangle \qquad \underset{\sim}{U}|u_j\rangle = u_j|u_j\rangle$$

when $u_i \neq u_j$

when $i \neq j$ (Degeneracy as for Hermitian).

$$\langle u_j|\underbrace{\underset{\sim}{U}^+\underset{\sim}{U}}_{\underset{\sim}{\mathbb{1}}}|u_i\rangle = u_j^* u_i \langle u_j|u_i\rangle$$

$$\langle u_j|u_i\rangle = u_j^* u_i \langle u_j|u_i\rangle$$

$$(1 - u_j^* u_i)\langle u_j|u_i\rangle = 0$$

1) $u_i \neq u_j$, $i \neq j$: $(1 - u_j^* u_i) \neq 0$ $\therefore \langle u_j|u_i\rangle = 0$

2) $i = j$, $u_i = u_j$, $\langle u_i|u_i\rangle = 1$: $|u_i|^2 = 1$, $u_i = e^{i\theta_i}$

Simultaneous Diagonalization

Th'm 13: If $\underset{\sim}{\Omega}$ and $\underset{\sim}{\Lambda}$ commute, a basis of common eigenvectors that diagonalize them both. And vice versa.

Vice versa: $\underset{\sim}{\Omega} \doteq \begin{pmatrix} \omega_1 & & 0 \\ 0 & \omega_2 & \\ 0 & 0 & \ddots \\ \vdots & & & \omega_n \end{pmatrix}$ $\underset{\sim}{\Lambda} \doteq \begin{pmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & \\ 0 & & \ddots & \\ \vdots & & & \lambda_n \end{pmatrix}$

$$\underset{\sim}{\Omega}\underset{\sim}{\Lambda} \doteq \begin{pmatrix} \omega_1\lambda_1 & 0 & \cdots & \\ 0 & \omega_2\lambda_2 & & \\ \vdots & & \ddots & \\ & & & \omega_n\lambda_n \end{pmatrix} = \begin{pmatrix} \lambda_1\omega_1 & 0 & & \\ 0 & \lambda_2\omega_2 & & \\ \vdots & & \ddots & \\ & & & \lambda_n\omega_n \end{pmatrix} = \underset{\sim}{\Lambda}\underset{\sim}{\Omega}$$

Primary: $\underset{\sim}{\Omega}|\omega_i\rangle = \omega_i|\omega_i\rangle$ (assume $\{\omega_i\}$ non degenerate)

$$\underbrace{\underset{\sim}{\Lambda}\underset{\sim}{\Omega}}_{\text{commute}}|\omega_i\rangle = \omega_i\underset{\sim}{\Lambda}|\omega_i\rangle$$

$$\underset{\sim}{\Omega}\left(\underset{\sim}{\Lambda}|\omega_i\rangle\right) = \omega_i\left(\underset{\sim}{\Lambda}|\omega_i\rangle\right) \Rightarrow \underset{\sim}{\Lambda}|\omega_i\rangle \propto |\omega_i\rangle$$

or, $|\omega_i\rangle$ also an eigenket of $\underset{\sim}{\Lambda}$!

What about degeneracy in $\underset{\sim}{\Omega}$?
easy... inside the degenerate subspace, ANY BASIS keeps $\underset{\sim}{\Omega}$ diagonal. So one diagonalizes $\underset{\sim}{\Lambda}$ inside the degenerate subspace of $\underset{\sim}{\Omega}$.

$\underset{\sim}{\Lambda}$ may not be similarly degenerate!

For example, maybe ...

$$\underset{\sim}{\Omega} \doteq \begin{pmatrix} w_1 & 0 & 0 \\ 0 & w_2 & 0 \\ 0 & 0 & w_2 \end{pmatrix} \qquad \underset{\sim}{\Lambda} \doteq \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$$

note in this case, the three states can be uniquely "tagged"

$(w_1, \lambda_1) \Longrightarrow$ state # 1

$(w_2, \lambda_1) \Longrightarrow$ state # 2

$(w_2, \lambda_2) \Longrightarrow$ state # 3

when this occurs, the set $\{\underset{\sim}{\Omega}, \underset{\sim}{\Lambda}\}$ is called a "complete set of commuting operators"

1 $\underset{\sim}{\Omega}$ could be a CSCO if totally non-degenerate.

---

skip 1.8

Functions of operators

defined by the power series

if $\quad f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad$ z is called a "commuting number" or c-number.

then $\quad f(\underset{\sim}{\Omega}) = \sum_{n=0}^{\infty} a_n \underset{\sim}{\Omega}^n \qquad$ $\underset{\sim}{\Omega}$ sometimes called a q-number

this is particularly easy when one works in the eigenbasis of $\underset{\sim}{\Omega}$:

$$\underset{\sim}{\Omega} \doteq \begin{pmatrix} \omega_1 & & & 0 \\ & \omega_2 & & \\ & & \ddots & \\ 0 & & & \omega_m \end{pmatrix}$$

$$f(\underset{\sim}{\Omega}) \doteq \begin{pmatrix} \sum_{n=0}^{\infty} a_n \omega_1^n & & & 0 \\ & \sum_{n=0}^{\infty} a_n \omega_2^n & & \\ & & \ddots & \\ 0 & & & \sum_{n=0}^{\infty} a_n \omega_m^n \end{pmatrix}$$

$$\doteq \begin{pmatrix} f(\omega_1) & 0 & & 0 \\ 0 & f(\omega_2) & & \\ 0 & & \ddots & \\ \vdots & & & f(\omega_m) \end{pmatrix}$$

---

example: $\underset{\sim}{\sigma}_1 \doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \underset{\sim}{\sigma}_1^2 \doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

so $e^{-i\theta \underset{\sim}{\sigma}_1} = \underset{\sim}{\mathbb{1}} - i\theta \underset{\sim}{\sigma}_1 + \frac{1}{2}(-i\theta)^2 \underbrace{\underset{\sim}{\sigma}_1^2}_{\underset{\sim}{\mathbb{1}}} + \frac{1}{3!}(-i\theta)^3 \underbrace{\underset{\sim}{\sigma}_1^3}_{\underset{\sim}{\sigma}_1} + \cdots$

$$= \underset{\sim}{\mathbb{1}}\left(1 - \frac{1}{2}\theta^2 + \frac{1}{4!}\theta^4 + \cdots\right)$$

$$- i\underset{\sim}{\sigma}_1\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \cdots\right)$$

$$\boxed{e^{-i\theta \underset{\sim}{\sigma}_1} = \underset{\sim}{\mathbb{1}} \cos\theta - i \underset{\sim}{\sigma}_1 \sin\theta}$$

also works for: $\underset{\sim}{\sigma}_2 \doteq \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\underset{\sim}{\sigma}_3 \doteq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

derivative:

$$\frac{d}{d\theta}\left(e^{-i\theta \underset{\sim}{\sigma}_1}\right) = \underset{\sim}{\mathbb{1}}(-\sin\theta) - i \underset{\sim}{\sigma}_1 \cos\theta \quad \text{(easy)}$$

Integral w/r to parameter is easy

what gets hard:

$$e^{\underset{\sim}{\Omega}} e^{\underset{\sim}{\Lambda}} \text{ generally } \neq e^{\underset{\sim}{\Omega}+\underset{\sim}{\Lambda}}$$

when $[\underset{\sim}{\Omega}, \underset{\sim}{\Lambda}] = 0$ then $e^{\underset{\sim}{\Omega}} e^{\underset{\sim}{\Lambda}} = e^{\underset{\sim}{\Omega}+\underset{\sim}{\Lambda}}$

why?

$$e^{\underset{\sim}{\Omega}} e^{\underset{\sim}{\Lambda}} = \underbrace{\left(1 + \underset{\sim}{\Omega} + \frac{1}{2!}\underset{\sim}{\Omega}^2 + \ldots\right)}_{\text{all } \underset{\sim}{\Omega}\text{'s to left of}} \underbrace{\left(1 + \underset{\sim}{\Lambda} + \frac{1}{2!}\underset{\sim}{\Lambda}^2 + \ldots\right)}_{\text{all } \underset{\sim}{\Lambda}\text{'s !}}$$

$$e^{(\underset{\sim}{\Omega}+\underset{\sim}{\Lambda})} = 1 + (\underset{\sim}{\Omega}+\underset{\sim}{\Lambda}) + \frac{1}{2!}(\underset{\sim}{\Omega}+\underset{\sim}{\Lambda})(\underset{\sim}{\Omega}+\underset{\sim}{\Lambda}) + \ldots$$

this $\underset{\sim}{\Lambda}$ is to right of THIS $\underset{\sim}{\Omega}$