Adjoints

Recall a linear operator, abstractly called $\underset{\sim}{\Omega}$, can be represented or imaged in a particular and specific basis $(|1\rangle, |2\rangle, \ldots, |n\rangle)$ or $\{|i\rangle\}$

$$\underset{\sim}{\Omega} \doteq \begin{pmatrix} \Omega_{11} & \Omega_{12} & \cdots & \Omega_{1n} \\ \Omega_{21} & \ddots & & \\ \vdots & & \ddots & \\ \Omega_{n1} & & & \Omega_{nn} \end{pmatrix} = \begin{pmatrix} \langle 1|\underset{\sim}{\Omega}|1\rangle & \langle 1|\underset{\sim}{\Omega}|2\rangle & \cdots & \langle 1|\underset{\sim}{\Omega}|n\rangle \\ \langle 2|\underset{\sim}{\Omega}|1\rangle & & & \\ \vdots & & \ddots & \\ \langle n|\underset{\sim}{\Omega}|1\rangle & & & \langle n|\underset{\sim}{\Omega}|n\rangle \end{pmatrix}$$

A "matrix element" of $\underset{\sim}{\Omega}$ means a "bracket" (think braΩket!) between the bra labeled i and the ket labeled j:

$$\Omega_{ij} = \langle i|\underset{\sim}{\Omega}|j\rangle$$

Another way to view the operator $\underset{\sim}{\Omega}$ is to use the <u>completeness</u> of the basis:

completeness: $\sum_i |i\rangle\langle i| = \sum_j |j\rangle\langle j| = \underset{\sim}{\mathbb{1}}$

just different label

$$\underset{\sim}{\Omega} = \underset{\sim}{\mathbb{1}} \cdot \underset{\sim}{\Omega} \cdot \underset{\sim}{\mathbb{1}} = \left(\sum_i |i\rangle\langle i|\right) \underset{\sim}{\Omega} \left(\sum_j |j\rangle\langle j|\right)$$

$$= \sum_{ij} |i\rangle\langle i|\underset{\sim}{\Omega}|j\rangle\langle j|$$

$$\boxed{\underset{\sim}{\Omega} = \sum_{ij} |i\rangle \Omega_{ij} \langle j|}$$

Right Hand Side obviously basis dependent - basis bra's & kets are involved!

to take adjoint....

$$\underset{\sim}{\Omega} \doteq \begin{pmatrix} \Omega_{11} & \Omega_{12} & \cdots & \Omega_{1n} \\ \Omega_{21} & \Omega_{22} & & \\ \vdots & & \ddots & \\ \Omega_{n1} & & & \Omega_{nn} \end{pmatrix} ; \quad \underset{\sim}{\Omega}^{+} = \begin{pmatrix} \Omega_{11}^{*} & \Omega_{21}^{*} & & \Omega_{n1}^{*} \\ \Omega_{12}^{*} & \Omega_{22}^{*} & & \\ & & \ddots & \\ \Omega_{1n}^{*} & & & \Omega_{nn}^{*} \end{pmatrix}$$

(flipped across diagonal + c.c.)

$$= \sum_{ij} |i\rangle \Omega_{ij} \langle j| \qquad \underset{\sim}{\Omega}^{+} = \sum_{ij} |i\rangle \Omega_{ji}^{*} \langle j|$$

($|i\rangle\langle j|$ goes with spot on matrix)

$$\underset{\sim}{\Omega}^{+} = \sum_{ij} |j\rangle \Omega_{ij}^{*} \langle i|$$

rules: • mirror image bras + kets
• complex conjugate numbers

ket: $\underset{\sim}{\Omega}|V\rangle = \underset{\sim}{\mathbb{1}}\,\underset{\sim}{\Omega}\,\underset{\sim}{\mathbb{1}}|V\rangle$

$$= \sum_{ij} |i\rangle \underbrace{\langle i|\underset{\sim}{\Omega}|j\rangle}_{\Omega_{ij}} \underbrace{\langle j|V\rangle}_{V_j}$$

Ω_{ij} matrix element of $\underset{\sim}{\Omega}$ — V_j = component of $|V\rangle$ — both basis dependent

$$\underset{\sim}{\Omega}|V\rangle = \sum_{ij} |i\rangle \underbrace{\Omega_{ij} V_j}_{\text{matrix times vector}}$$

or: $\underset{\sim}{\Omega}|V\rangle \doteq \begin{pmatrix} \sum_j \Omega_{1j} V_j \\ \sum \Omega_{2j} V_j \\ \vdots \\ \sum \Omega_{nj} V_j \end{pmatrix}$ now <u>dual</u> of this ?

dual of $\begin{pmatrix} \sum_j \Omega_{1j} v_j \\ \sum_j \Omega_{2j} v_j \\ \vdots \\ \sum_j \Omega_{nj} v_j \end{pmatrix}$ is $\left(\sum_j \Omega^*_{1j} v^*_j \quad \sum_j \Omega^*_{2j} v^*_j \quad \cdots \quad \sum_j \Omega^*_{nj} v^*_j \right)$

Two Views

Matrix

$$\underbrace{(v_1^* \; v_2^* \; \cdots \; v_n^*)}_{\langle v|} \underbrace{\begin{pmatrix} \Omega^*_{11} & \Omega^*_{21} & \cdots & \Omega^*_{n1} \\ \Omega^+_{12} & \Omega^+_{22} & & \vdots \\ \Omega^+_{13} & & & \vdots \\ \vdots & & & \\ \Omega^+_{1n} & & \cdots & \Omega^+_{n1} \end{pmatrix}}_{\Omega^+}$$

Bra-Ket

$$\sum_{ij} \Omega^*_{ij} v^*_j \langle i|$$

$$v_j^* = \langle j|v\rangle^* = \langle v|j\rangle$$

$$\sum_{ij} \langle v|j\rangle \underbrace{\Omega^*_{ij} \langle i|}_{\Omega^+}$$

$$= \langle v| \Omega^+$$

The bottom line:

dual of $\Omega|v\rangle$ is $\langle v|\Omega^+$

note: $\Lambda \Omega |v\rangle$ is $\langle v|\Omega^+ \Lambda^+$

(not necessarily $\langle v|\Omega$)

when Ω does $= \Omega^+$, Ω called "Hermitian"

when Ω lives in one dimension, this means
$\Omega_{11} = \Omega^*_{11} \Rightarrow \Omega$ represented by a real number!

$\Omega = \Omega^+$ (Ω is Hermitian) means Ω sort of real

also: when $\Omega = -\Omega^+$, Ω called "anti-Hermitian" and it is kind of "imaginary"

Brief Physics Comment:

systems (like a particle, an atom, a bunch of atoms in a gas)
will be associated with kets & bras.

observables (like position, momentum, energy)
will be associated with Hermitian Operators

Unitary Operators:

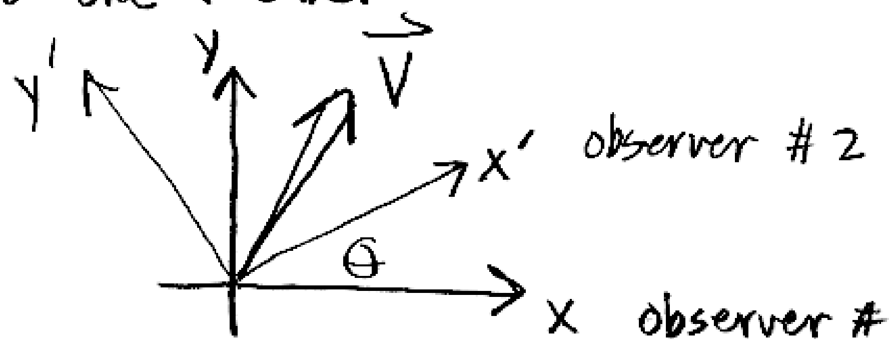
- the adjoint is the inverse
- rarely Hermitian (rarely observables)

$$\underset{\sim}{U}\,\underset{\sim}{U}^{\dagger} = \underset{\sim}{U}^{\dagger}\,\underset{\sim}{U} = \underset{\sim}{\mathbb{1}}$$

What good are unitary operators?

⇒ They allow reconcilliation of different, but equivalent, points of view.

⇒ for example, 2 observers may use different coordinate systems, simply rotated with respect to one & other:



⇒ if observer #1 reports that the vector $\vec{V}$ has components (V_x, V_y), these will differ from the components (V_x', V_y') reported by observer #2

$\Rightarrow$ but (V_x, V_y) and (V_x', V_y') will be related by a matrix that represents a unitary operator..

Ket description

$$\underset{\sim}{\mathbb{1}} = \sum_i |i\rangle\langle i| \quad \text{"completeness"}$$
$$= \sum_{i'} |i'\rangle\langle i'| \quad \text{"}$$

$$|V\rangle = \underset{\sim}{\mathbb{1}} |V\rangle$$

$$= \sum_{i=1}^{n} \underbrace{|i\rangle\langle i|}_{\text{observer \#1}} V\rangle = \sum_{i'=1}^{n} \underbrace{|i'\rangle\langle i'|}_{\text{observer \#2}} V\rangle$$

to jiggle this + make a unitary transformation drop out.... "bra"-in with $\langle j'|$

$$\langle j'|V\rangle = \sum_{i=1}^{n} \underbrace{\langle j'|i'\rangle}_{\delta_{j'i'}} \langle i'|V\rangle = \sum_{i=1}^{n} \underbrace{\langle j'|i\rangle}_{\substack{\text{not easy!}\\ \text{different bases}}} \langle i|V\rangle$$

$$\langle j'|V\rangle = \underbrace{\langle j'|V\rangle}_{\substack{\text{component}\\ \text{of V to observer}\\ \#2}} = \sum_{i=1}^{n} \langle j'|i\rangle \underbrace{\langle i|V\rangle}_{\substack{\text{component of V}\\ \text{to observer \#1}}}$$

matrix elements of a unitary transformation!

$$V_j' = \sum_{i=1}^{n} \langle j'|i\rangle V_i$$

Claim that $U_{ji} \equiv \langle j' | i \rangle$ are the elements of a unitary matrix.

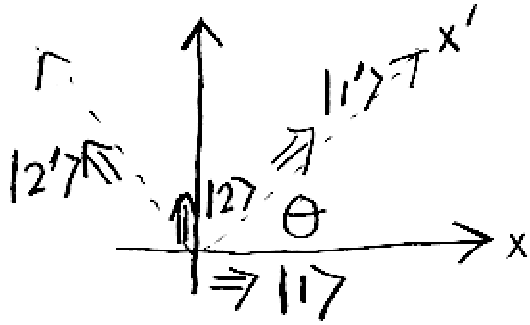
$$(U^\dagger)_{ji} = U_{ij}^* = \langle i' | j \rangle^* = \langle j | i' \rangle$$

then $\underset{\sim}{U}\underset{\sim}{U}^\dagger \doteq \sum_{k=1}^{n} U_{ik} U^\dagger_{kj} = \sum_{k=1}^{n} \langle i' | \underbrace{k \rangle\langle k}_{\underset{\sim}{1}\,!} | j' \rangle$

$$\sum_{k=1}^{n} U_{ik} U^\dagger_{kj} = \underbrace{\langle i' | j' \rangle}_{\text{same basis!}}$$

$$\underset{\sim}{U}\underset{\sim}{U}^\dagger \doteq \sum_{k=1}^{n} U_{ik} U^\dagger_{kj} = \delta_{ij} \doteq \underset{\sim}{1} \quad (\underset{\sim}{U}^\dagger \underset{\sim}{U} \text{ is homework})$$

"Show Me" for rotated x-y system.



$|1\rangle$ is $\hat{i}$ $\quad |2\rangle$ is $\hat{j}$

$|1'\rangle$ is $\hat{i}\cos\theta + \hat{j}\sin\theta$

$|2'\rangle$ is $-\hat{i}\sin\theta + \hat{j}\cos\theta$

$$\underset{\sim}{U} \doteq \begin{pmatrix} \langle 1'|1\rangle & \langle 1'|2\rangle \\ \langle 2'|1\rangle & \langle 2'|2\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

so $$\begin{pmatrix} V_x' \\ V_y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} V_x' \\ V_y' \end{pmatrix} \doteq |V\rangle = \underset{\sim}{U} |V'\rangle$$

and $$\underset{\sim}{U}^\dagger \underset{\sim}{U} \doteq \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \overbrace{\cos^2\theta + \sin^2\theta}^{1} & \overbrace{-\sin\theta\cos\theta + \sin\theta\cos\theta}^{0} \\ \overbrace{-\sin\theta\cos\theta + \sin\theta\cos\theta}^{0} & \overbrace{\sin^2\theta + \cos^2\theta}^{1} \end{pmatrix}$$

Preservation of Inner Product

when $|V_1'\rangle = \underset{\sim}{U}|V_1\rangle$

$$|V_2'\rangle = \underset{\sim}{U}|V_2\rangle$$

$$\langle V_2'|V_1'\rangle = \langle V_2|\underbrace{\underset{\sim}{U}^{\dagger}\underset{\sim}{U}}_{\underset{\sim}{\mathbb{1}}}|V_1\rangle = \langle V_2|V_1\rangle$$

Generalization: rotations don't alter dot product

Differing Representations of Operators

Observer #2: $\underset{\sim}{\Omega} \doteq \begin{pmatrix} \langle 1'|\underset{\sim}{\Omega}|1'\rangle & \langle 1'|\underset{\sim}{\Omega}|2'\rangle & \cdots & \langle 1'|\underset{\sim}{\Omega}|n'\rangle \\ \langle 2'|\underset{\sim}{\Omega}|1'\rangle & & & \\ \vdots & & \ddots & \\ \langle n'|\underset{\sim}{\Omega}|1'\rangle & & & \langle n'|\underset{\sim}{\Omega}|n'\rangle \end{pmatrix}$

$$= \sum_{jk} |j'\rangle \Omega'_{jk} \langle k'| \quad \text{where } \Omega'_{jk} = \langle j'|\underset{\sim}{\Omega}|k'\rangle$$

insert $\underset{\sim}{\mathbb{1}} = \sum_{i=1}^{n} |i\rangle\langle i|$ insert $\underset{\sim}{\mathbb{1}} = \sum_{l=1}^{n} |l\rangle\langle l|$

$$\underset{\sim}{\Omega} \doteq \sum_{ijkl} |i\rangle \underbrace{\langle i|j'\rangle \Omega'_{jk} \langle k'|l\rangle}_{} \langle l|$$

$$\boxed{\Omega_{il} = \sum_{jk} U^{\dagger}_{ij} \Omega'_{jk} U_{kl}}$$

The unitary transformation also tells how to turn one representation of an operator into another!

a second way to derive this relationship:

$$\Omega_{il} = \langle i|\underset{\sim}{\Omega}|l\rangle = \sum_{jk} \underbrace{\langle i|j'\rangle}_{U^{\dagger}_{ij}} \underbrace{\langle j'|\underset{\sim}{\Omega}|k'\rangle}_{\Omega'_{jk}} \underbrace{\langle k'|l\rangle}_{U_{kl}}$$

insert $\underset{\sim}{\mathbb{1}}$ inset $\underset{\sim}{\mathbb{1}}$

$\underset{\sim}{\Omega} = \underset{\sim}{U}^{\dagger}\underset{\sim}{\Omega}'\underset{\sim}{U}$ but "abstraction" of $\underset{\sim}{\Omega}$ is lost

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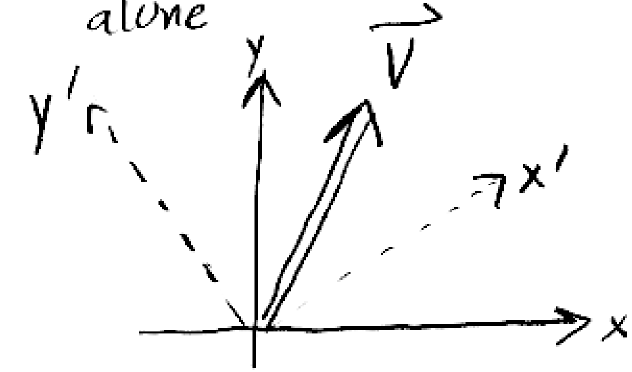
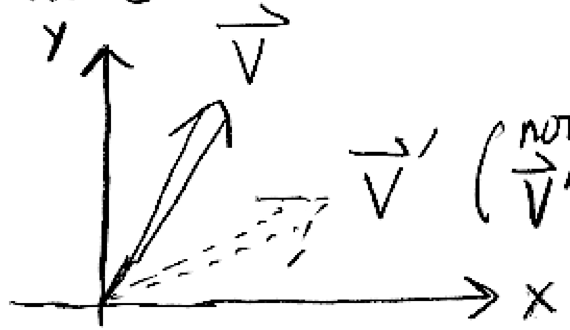
Viewpoints

"Passive" → change coordinates, leave vectors alone



(discussed so far).

"Active" → change vectors, leave coordinates alone.



(note, components of $\vec{V}'$ are same as components of $\vec{V}$ in x', y' above).

What about operators? Seems like <u>lots</u> of matrices can represent the same operator. What is fundamental, and what is variable?

<u>Fundamental</u>: Determinant, Trace, Eigenvalues

Recall: $\det(\underset{\sim}{\Lambda}\,\underset{\sim}{\Omega}) = (\det\underset{\sim}{\Lambda})\cdot(\det\underset{\sim}{\Omega})$

$$\det(\underset{\sim}{\Omega}) = \det(\underset{\sim}{U}^{+}\underset{\sim}{\Omega}'\underset{\sim}{U}) = (\det\underset{\sim}{U}^{+})\cdot(\det\underset{\sim}{U})\cdot\det\underset{\sim}{\Omega}')$$

but since $\underset{\sim}{U}^{+}\underset{\sim}{U} = \underset{\sim}{\mathbb{1}}$

$$(\det\underset{\sim}{U}^{+})(\det\underset{\sim}{U}) = \det(\underset{\sim}{\mathbb{1}}) = 1$$

$$\boxed{\det\underset{\sim}{\Omega} = \det\underset{\sim}{\Omega}'}$$ Unitary Transformations Don't change determinants

Recall: $\mathrm{Tr}(\underset{\sim}{A}\underset{\sim}{B}\underset{\sim}{C}) = \mathrm{Tr}(\underset{\sim}{C}\underset{\sim}{A}\underset{\sim}{B}) = \mathrm{Tr}(\underset{\sim}{B}\underset{\sim}{C}\underset{\sim}{A})$

so $\mathrm{Tr}(\underset{\sim}{\Omega}) = \mathrm{Tr}(\underset{\sim}{U}^{+}\underset{\sim}{\Omega}'\underset{\sim}{U}) = \mathrm{Tr}(\underset{\sim}{U}\underset{\sim}{U}^{+}\underset{\sim}{\Omega}')$

$$\boxed{\mathrm{Tr}(\underset{\sim}{\Omega}) = \mathrm{Tr}(\underset{\sim}{\mathbb{1}}\underset{\sim}{\Omega}') = \mathrm{Tr}(\underset{\sim}{\Omega}')}$$

Eigenvalues

$$\underset{\sim}{\Omega}|V\rangle = \underbrace{\omega|V\rangle}_{\text{Get multiple of the same ket back again!}}$$

List of operations

Most interesting case in physics: when $\underset{\sim}{\Omega}$ represents the operation of "hurtling the system forward in time" (a.k.a. $\underset{\sim}{H}$). If the system starts in a ket that is an eigenket of $\underset{\sim}{H}$, then the system is <u>stable</u>, although there may be "dynamic equilibrium"