Schwartz Inequality (Thm 5)

\[ |Kv|w\| \leq |v|w| \]

meaning... recall vectors \( \vec{A}, \vec{B} = \|A\||B|/\cos\theta \uparrow \)

\[ |\cos\theta| \leq 1 \]

so \( |A . \vec{B}| \leq |A| . |\vec{B}| \).

\[ \uparrow \text{equality when } \cos\theta = \pm 1 \]
then \( A \parallel \vec{B} \text{ or } A \perp \vec{B} \)

More generally...

consider... \( |v\rangle + \lambda |w\rangle = |z\rangle \)

\[ \uparrow \text{complex, arbitrary} \]

must be: \( \langle z | z \rangle \geq 0 \)

\[ \{ \langle v | + \lambda^* \langle w | \} \{ |v\rangle + \lambda |w\rangle \} \geq 0 \]

\[ \langle v |v\rangle + |\lambda|^2 \langle w |w\rangle + \lambda \langle v |w\rangle + \lambda^* \langle w |v\rangle \geq 0 \]

\[ \text{minimize left hand side} \]

\[ \lambda = \lambda_R + i \lambda_I \] (variables)

\[ \langle v |w\rangle = \alpha + i \beta \]

\[ \lambda \langle v |w\rangle + \lambda^* \langle w |v\rangle = (\lambda_R + i \lambda_I)(\alpha + i \beta) + (\lambda_R - i \lambda_I)(\alpha - i \beta) \]

\[ = 2\alpha \lambda_R - 2\beta \lambda_I \]

\[ |\lambda|^2 = \lambda_R^2 + \lambda_I^2 \]
left hand side:
\[ f(V) = \langle V|V \rangle + (\lambda_r^2 + \lambda_i^2) \langle W|W \rangle + 2 \alpha \lambda_r - 2 \beta \lambda_i \]

differentiate, set = 0

\[ \frac{\delta f}{\delta \lambda_r} = 2 \lambda_r \langle W|W \rangle + 2 \alpha = 0 \quad \frac{\delta f}{\delta \lambda_i} = 2 \lambda_i \langle W|W \rangle - 2 \beta = 0 \]

\[ \lambda_r = -\frac{\alpha}{\langle W|W \rangle} \quad \lambda_i = \frac{\beta}{\langle W|W \rangle} \]

\[ \lambda = \lambda_r + i \lambda_i = -\alpha + i \beta \quad = -\frac{\langle V|W \rangle^*}{\langle W|W \rangle} = -\frac{\langle W|V \rangle}{\langle W|W \rangle} \]

this \( \lambda \) is a minimum because

\[ \frac{\delta^2 f}{\delta \lambda_r^2} = 2 \langle W|W \rangle > 0 \quad \frac{\delta^2 f}{\delta \lambda_i^2} = 2 \langle W|W \rangle > 0 \]

\[ \frac{\delta^2 f}{\delta \lambda_r \delta \lambda_i} = \frac{\delta^2 f}{\delta \lambda_i \delta \lambda_r} = 0 \]

so \[ |Z\rangle = |V\rangle - \frac{\langle W|V \rangle}{\langle W|W \rangle} \]

\[ = |V\rangle - \frac{|W\rangle \langle W|V \rangle}{\langle W|W \rangle} \]

is the shortest \( |Z\rangle \) imaginable and it still must be greater than or equal to zero.

\[ \langle Z|Z \rangle = (\langle V| - \frac{\langle V|W \rangle}{\langle W|W \rangle} \langle W|) (|V\rangle - \frac{\langle W|V \rangle}{\langle W|W \rangle} |W\rangle) > 0 \]
\[ \langle v|v \rangle - \frac{\langle v|w \rangle \langle w|v \rangle}{\langle w|w \rangle} - \frac{\langle w|v \rangle \langle v|w \rangle}{\langle w|w \rangle} + \frac{\langle v|w \rangle \langle w|v \rangle}{\langle w|w \rangle} \geq 0 \]

\[ \langle v|v \rangle \leq 1 \]

\[ |\langle v|w \rangle|^2 \leq |v|^2 |w|^2 \]

\[ |\langle v|w \rangle| \leq |v||w| \]

QED.

Very Important for the uncertainty principle.

ISA Nelson
**Subspaces**

D11: Given \( V \), a set of elements that form a vector space among themselves is called a subspace. A particular subspace of dimensionality 1 is called \( V_i^{ni} \).

Examples: in 3-space \((x, y, z)\) vectors with only \( x \)-components are a subspace \((V_x^1)\).

Vectors with only \( x-y \) components (those in the \( x-y \) plane) form a subspace \((V_{xy}^2)\).

D11: about "breaking down" a vector space.

What about building up a vector space?

D12: Given two subspaces \( V_i^{ni} \) and \( V_j^{mj} \), \( V_i^{ni} \oplus V_j^{mj} \), the sum, is defined as the set:

1. all elements of \( V_i^{ni} \)
2. all elements of \( V_j^{mj} \)
3. all possible linear combinations of elements of \( V_i^{ni} \) and \( V_j^{mj} \).

Examples: \( V_x^1(R) \) and \( V_y^1(R) \) has all vectors of the form: \( ax \), \( by \), \( ax + by \) don't forget #3.

![Diagram](image1.png)

#1

![Diagram](image2.png)

#2

![Diagram](image3.png)

#3
as you should expect, the dimensionality of \( W_i^n \oplus W_j^m \) is, when all elements of \( W_i^n \) are orthogonal to all elements of \( W_j^m \), equal to \( n_i + m_j \).

\[
V_{xy}^2 \oplus V_{xz}^2 = 4 \quad \text{? not 4 because both vector spaces include x so 3 is the answer}
\]

Addition of spaces is important!

**Linear Operators**

Operator: a list of instructions, to do to a ket (abstract vector) \( |V\rangle \) and arrive at some other ket \( |V'\rangle \)

Linear Operator: when the initial kets have a linear relationship, then the final kets have that same linear relationship.

\[
\Omega |V\rangle = |V'\rangle
\]

Linear Operator: when \( |U\rangle = a |V\rangle \)

\[
\Omega |U\rangle = |V'\rangle
\]

then \( |U'\rangle = a |V'\rangle \)

aka \( \Omega a |V\rangle = a \Omega |V\rangle \)
more generally, when \( |z\rangle = \alpha |v\rangle + \beta |w\rangle \)
then for a linear operator \( \Omega \)
\( \Omega |z\rangle = \alpha \Omega |v\rangle + \beta \Omega |w\rangle \)

Operators can also operate on the duals of vectors, that is, on bras as well as kets:

\[ \langle w | \Omega = \langle w | \Omega \]

since \( \Omega \) a list of instructions, no problem

**Linear operator:**

\[ (\alpha \langle w | + \beta \langle z |) \Omega = \alpha \langle w | \Omega + \beta \langle z | \Omega \]

**Examples:** \( \Omega = I \) or \( \Omega \)

leaves vector alone!

linear.

In xyz space \( R(\frac{\pi}{2}, \hat{z}) \)

read: rotate by \( \frac{\pi}{2} \)
counter-clockwise

about \( \hat{z} \)

[General case: \( R(\hat{\theta}) \) means rotate counterclockwise by \( |\hat{\theta}| \) about the \( \hat{\theta} \) direction]

\[ R(\frac{\pi}{2}, \hat{z}) |1\rangle = |1\rangle \]
\[ R(\frac{\pi}{2}, \hat{z}) |2\rangle = |3\rangle \]
\[ R(\frac{\pi}{2}, \hat{z}) |3\rangle = -|2\rangle \]

Linear? From figure it is (not exactly obvious).
Only need to evaluate action of operator \( \Omega \) on a basis... preferably an orthonormal basis.

\[ \Omega |i\rangle = |i'\rangle \]

Then if \( |v\rangle = \sum_{i=1}^{n} v_i |i\rangle \)

\[ \Omega |v\rangle = \sum_{i=1}^{n} v_i \Omega |i\rangle = \sum_{i=1}^{n} v_i |i'\rangle \]

That is, the components of \( |v\rangle \) in the basis \( |i\rangle \) will be the same as the components of \( |v'\rangle = \Omega |v\rangle \) in the basis \( |i'\rangle \), where \( |i'\rangle = \Omega |i\rangle \)

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**OPERATORS MAY NOT COMMUTE!**

\( \Omega, \Lambda \) are two operators (possibly linear)

\[ \Lambda (\Omega |v\rangle) = \Lambda \Omega |v\rangle \]  
means do \( \Omega \)'s list  
and then do \( \Lambda \)'s

\[ \Omega (\Lambda |v\rangle) = \Omega \Lambda |v\rangle \]  
means do \( \Lambda \)'s list  
and then do \( \Omega \)'s

Often \[ \Lambda \Omega |v\rangle \neq \Omega \Lambda |v\rangle \]

we say: \[ \Omega \Lambda - \Lambda \Omega \neq 0 \]

called the commutator of \( \Omega \) and \( \Lambda \)

notation: \[ [\Omega, \Lambda] = \Omega \Lambda - \Lambda \Omega \]
Example: in $S$-space, $N_{x'y'z'}(R)$

\[
R(\frac{1}{2}\pi \hat{i}) R(\frac{1}{2}\pi \hat{j}) \neq R(\frac{1}{2}\pi \hat{j}) R(\frac{1}{2}\pi \hat{i})
\]

will do this as a demonstration on a book:

Inverses $\frac{\sigma}{\nu}$ doesn't always have an inverse.

When $\frac{\sigma}{\nu} |\psi\rangle = 0$ and $|\psi\rangle \neq 0$, no inverse.

\[
\frac{\sigma}{\nu} \frac{\sigma}{\nu}^{-1} = I = \frac{\nu}{\sigma} \frac{\sigma}{\nu}
\]

\[
(\frac{\sigma}{\nu} \Lambda)^{-1} = \Lambda^{-1} \frac{\nu}{\sigma}^{-1}
\]

order reversed

\[
\frac{\sigma}{\nu} \Lambda \left( \Lambda^{-1} \frac{\nu}{\sigma}^{-1} \right) = I
\]
Matrix Elements

\[ |v\rangle \leftrightarrow \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \]

"project \(|v\rangle\) onto an orthonormal basis"

that is \( |v\rangle = \sum_{i=1}^{n} \langle i | v \rangle |i\rangle \)

\[ = \sum_{i=1}^{n} \langle i | v \rangle |i\rangle \]

Linear Operators are matrices

\[ \Omega \leftrightarrow \begin{pmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \cdots & \Omega_{1n} \\ \Omega_{21} & \Omega_{22} & \cdots & \cdots & \Omega_{2n} \\ \Omega_{31} & \Omega_{32} & \cdots & \cdots & \Omega_{3n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \Omega_{n1} & \Omega_{n2} & \cdots & \cdots & \Omega_{nn} \end{pmatrix} \]

"image or representation of \( \Omega \)"

The matrix representation **depends on the basis**, which means, the same \( \Omega \) can be represented by two different matrices! Given the basis \(|1\rangle, |2\rangle, \ldots, |n\rangle\)

\[ \Omega_{ij} = \langle i | \Omega | j \rangle \neq \langle i' | \Omega | j' \rangle \text{ (other basis)} \]
Example: \( R(\frac{1}{2}\pi \hat{z}) \): use \( xyz \) basis

\[
\begin{align*}
R(\frac{1}{2}\pi \hat{z}) 1\rangle &= 1\rangle \\
R(\frac{1}{2}\pi \hat{z}) 2\rangle &= 13\rangle \\
R(\frac{1}{2}\pi \hat{z}) 3\rangle &= -12\rangle \\
\langle 1 | R(\frac{1}{2}\pi \hat{z}) | 1 \rangle &= \langle 1 | 1 \rangle = 1 \\
\langle 1 | R | 2 \rangle &= \langle 1 | 3 \rangle = 0 \\
\langle 1 | R | 3 \rangle &= -\langle 1 | 2 \rangle \\
\langle 2 | R(\frac{1}{2}\pi \hat{z}) | 1 \rangle &= \langle 2 | 1 \rangle = 0 \\
\langle 2 | R | 2 \rangle &= \langle 2 | 3 \rangle = 0 \\
\langle 2 | R | 3 \rangle &= -\langle 2 | 1 \rangle \\
\langle 3 | R(\frac{1}{2}\pi \hat{z}) | 1 \rangle &= \langle 3 | 1 \rangle = 0 \\
\langle 3 | R | 2 \rangle &= \langle 3 | 3 \rangle = 1 \\
\langle 3 | R | 3 \rangle &= -\langle 3 | 2 \rangle = 0
\end{align*}
\]

\[
R(\frac{1}{2}\pi \hat{z}) \cong \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}
\]

But I could have labelled...

\[
R(\frac{1}{2}\pi \hat{z}) \cong \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

in this basis...

worse, could have picked a basis not parallel to the \( x, y \) directions!