Mathematical Introduction (Shankar, Chap. 1)

Abstraction of the concept of the "vector" and "vector" generally taught in lower division as a "magnitude" and a "direction" very physical definition.

eg: 1 dimension

\[ \begin{array}{c}
\text{negative} \\
-2x_1
\end{array} \quad \begin{array}{c}
0 \\
\text{addition}
\end{array} \quad \begin{array}{c}
2x_1 \\
\text{positive direction}
\end{array} \quad \begin{array}{c}
x \\
\text{positive}
\end{array} \]

\[ \vec{x}_1 = 3 \text{ in } +x \text{ direction} \]

can multiply by, say 2 \: \text{ twice as long}

most interesting: multiplication by a negative number reverses direction

Mathematicians have "boiled" down the features that are important for defining the "linear vector space" and the "field" of numbers which can multiply the vectors

\[ \rightarrow \text{ page 2 of Shankar} \]

\[ \rightarrow \text{ Why is this activity important for quantum mechanics?} \]
Answer #1

\[ \psi = a_{100}^* + a_{200}^* + a_{210} + \ldots \]

Hydrogen atom...

"arbitrary state"

\[ e^2 \]

\[ n = 2 \]
\[ l = 0 \]
\[ m = 0 \]

\[ a_{21-1} \]
\[ n = 2 \]
\[ l = 1 \]
\[ m = 1, 0, -1 \]

\[ \begin{pmatrix} a_{100} \\ a_{200} \\ a_{211} \\ a_{210} \\ a_{21-1} \end{pmatrix} \]

\( \text{vector with many components} \)

\( \text{meaning of number is } "\text{amount}" \text{ of a given state in the arbitrary state."} \)

Answer #2: SPIN

\( \text{spin-} \frac{1}{2} : \quad \Psi = a \uparrow + b \downarrow \quad 2-d \)

spin-up spin-down

but... a + b must be complex numbers.

\( \text{spin-1: } \quad \Psi = a \uparrow + b \Rightarrow + c \downarrow \quad 3-d \)
\( \quad - \frac{3}{2} \quad \Psi = a \uparrow + b \Rightarrow + c \downarrow + d \downarrow \quad 4-d \)
Answer #3: Wave functions

\[ \Psi(x) \]

\[ \Psi(x_1) \rightarrow \] 

\[ 0, x_1, x_2, x_3, \text{etc.} \]


\[ \begin{bmatrix} \Psi(x_1) \\ \Psi(x_2) \\ \Psi(x_3) \end{bmatrix} \rightarrow \text{vector with an infinite number of dimensions} \]

Interesting concept \( \leftrightarrow \) "dimensionality" of the vector space.

Notation: not \( \vec{v} \) (implies 2-d or 3-d), but \( |v\rangle \) say "ket \( v \)"

- could have \( \infty \) dimensions
- satisfies rules on page 2

\[ |v\rangle + |-v\rangle = 0 \quad \text{and} \quad |-v\rangle = (-1) \cdot |v\rangle \]

inverse under addition
\( |v \rangle \) is "abstract" but can be "imagined" or "represented" by organizing its components in a vector, matrix, or function depending upon dimensionality, might have

\[ |v \rangle \leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{or} \quad |v \rangle \leftrightarrow \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \quad \text{or} \quad |v \rangle \leftrightarrow \psi(x) \]

to deal with dimensionality, first need concept of linear dependence & independence.

Concept: think 2 dimensions

\[
\begin{align*}
\vec{a} & \quad \vec{b} & \quad \vec{c} & \quad \vec{d} \\
\begin{cases}
\text{all linearly dependent (parallel)} \\
\end{cases}
\end{align*}
\]

why? \( \vec{b} = \text{(number)} \times \vec{a} \) et. cetera

but \( \vec{a} \uparrow \vec{b} \)

\( \vec{\lambda} \) and \( \vec{\beta} \) are linearly independent because they are not parallel.

\( \overrightarrow{\beta} = (\text{number}) \times \overrightarrow{\alpha} \)

\( \overrightarrow{\beta} - (\text{number}) \times \overrightarrow{\alpha} = 0 \)

\( \overrightarrow{\alpha} \), \( \overrightarrow{\beta} + \overrightarrow{\alpha} = 0 \)

\( \overrightarrow{\alpha} \), \( \overrightarrow{\beta} \)
Definition: given n vectors

\[ \sum_{i=1}^{n} a_i |i\rangle = 0 \]

The only way is \( a_1 = a_2 = \ldots = a_n = 0 \) for real numbers.

Say

\[ |1\rangle = (0 \ 1) \quad |2\rangle = (0 \ 1) \quad |3\rangle = (-2 \ -1) \]

Are these linearly independent?

Note: \(-2|2\rangle = \begin{pmatrix} -2 & -2 \\ 0 & -2 \end{pmatrix}\)

\[-2|2\rangle + |1\rangle = \begin{pmatrix} -2 & -1 \\ 6 & -2 \end{pmatrix}\]

So \( |1\rangle - 2|2\rangle - |3\rangle = \begin{pmatrix} 0 & 0 \end{pmatrix} = 0 \)

Not linearly independent (Exercise 1.14).
Definition: A vector space has dimension n if it can accommodate a maximum of n linearly independent vectors.

\[ V^n(R) \leq n \text{ dimensional} \]
\[ V^n(C) \leq n \text{ dimensional} \]

#4 in book p.5

\[ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ is 4 dimensional} \]
\[ \text{(even if } a, b, c, d \text{ complex)} \]

pretty clear

\[ \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \]

\[ \text{the 4 } \text{i} \text{.i. vectors} \]
\[ \begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 2 \end{bmatrix}, \begin{bmatrix} 3 \end{bmatrix}, \begin{bmatrix} 4 \end{bmatrix} \]

Thm #1: any vector \( |V\rangle \) in an n-dim. space can be written as a linear combination of n linearly independent vectors \( |1\rangle, \ldots, |n\rangle \).

Proof: if \( |V\rangle = \sum_{i=1}^{n} v_i |i\rangle \) for any set of \( v_i \),

then \( |V\rangle - \sum_{i=1}^{n} v_i |i\rangle \neq 0 \)
and there are \( n+1 \) linearly independent vectors... this contradicts the fact that the dimensionality was \( n \).

**Definition:** A set of \( n \) linearly independent vectors in an \( n \)-dimensional space is called a basis (§5 in book, p. 6).

**Comment:** NEED NOT BE ORTHOGONAL.

\[
\begin{align*}
\vec{a} &\quad \text{in 2-d, these are a basis!} \\
\vec{b} &
\end{align*}
\]

**Definition:** The coefficients of expansion \( V_i \) of a vector in terms of a basis \( \{i\} \) are called the components of the vector in that basis.

\[(*) \quad |V\rangle = \sum_{i=1}^{n} v_i |i\rangle\]

could imagine as

"imaging" or "representing" \( |V\rangle \)
Theorem; (*) is unique.

Proof: \( |\psi\rangle = \sum_{i=1}^{n} v_i |i\rangle = \sum_{i=1}^{n} v'_i |i\rangle \)

then \( \sum_{i=1}^{n} (v_i - v'_i) |i\rangle = 0 \)

basis!

linearly independent!

\( \therefore \) all \( v_i - v'_i = 0 \)
\( v_i = v'_i \)

also: if \( |\psi\rangle = \sum_{i=1}^{n} v_i |i\rangle \)

\( |\psi\rangle = \sum_{i=1}^{n} w_i |i\rangle \) components are sum of others

then \( |\psi\rangle + |\psi\rangle = \sum_{i=1}^{n} (v_i + w_i) |i\rangle \)

(can use page 2 to go through)

also \( a |\psi\rangle = \sum_{i=1}^{n} a v_i |i\rangle \)

components are multiples of old components