Physics 21 Problem Set 2

Harry Nelson

due Tuesday, Jan. 18 at 5pm

Course Announcements: The reading for this problem set is KK Chap. 4, pp. 152-173, and RHK4 Chapters 7 and 8. We will follow RHK4 in lecture.

---

1. Let's look in detail of the power of potential energy in solving challenging problems. We'll start by looking at the simple harmonic oscillator, where the potential energy is $U(x) = (1/2)kx^2$, where x is the displacement from static equilibrium of a particle of mass m, and k is the spring constant. Assume that the particle is displaced in the negative direction, to $-x_0$ where $x_0 > 0$, and then released. Don't explicitly use the formulas you've learned for motion of a simple harmonic oscillator; use energy concepts to describe the motion as outlined below. You'll material starting on p. 154 of KK and p. 160 of RHK4 useful in working this problem.

   (a) What is the energy E in the oscillator as a function of k and x_0? (This is pretty easy!)

   (b) Assume from here forward for numerical answers that $m = (1/20)\,\mathrm{kg}$, that $k = (\pi^2/20)\,\mathrm{kg/s^2}$, and that $x_0 = 3\,\mathrm{cm}$. Numerically evaluate E and give it in scientific notation, in Joules. You should get some number times 10^{-4} Joules. Make an accurate numerical plot of $U(x)$ for a domain of $x = -5\,\mathrm{cm}$ to $x = 5\,\mathrm{cm}$.

   (c) The particle's velocity as a function of x can be written in the form $v(x) = v_{\max} f([x/x_0])$. Give expressions for $v_{\max}$ in terms of m, k, and x_0, and for $f([x/x_0])$. When $x = \pm x_0$, that is, at the turning points, what is the value of $f([x/x_0])$?

   (d) Numerically evaluate $v_{\max}$ in m/s for the numbers provided. Make an accurate numerical plot of $v(x)$ for a domain of $x = -5\,\mathrm{cm}$ to $x = 5\,\mathrm{cm}$ for the numbers provided.

   (e) Think about the time it takes for the mass to move from $x = -x_0$ to $x = +x_0$. You can get an interesting lower bound, $T_{\min}$, on that time by simply assuming that the particle moves for the entire journey at $v(x) = v_{\max}$. Solve for $T_{\min}$ in terms of m, k, and other constants, and get a numerical result for the numbers provided.

   (f) The assumption that $v(x) = v_{\max}$ can be plotted on the numerical plot of part 1d; do so with a dashed line. Additionally, the assumption $v(x) = v_{\max}$ corresponds to a new and different potential energy, $U_1(x)$, that is not hard to deduce. Deduce $U_1(x)$, and plot it with a dotted line on the plot of part 1b.

   (g) Show that the exact time T to go from $x = -x_0$ to $x = x_0$ can be written as:

   $$T = \int_{-x_0}^{x_0} \frac{dx}{v(x)} = \frac{x_0}{v_{\max}} \int_{-1}^{1} \frac{dy}{f(y)}$$

   where the function $f(y)$ is the same function found in part 1c. What is the numerical value of the integral:

   $$\int_{-1}^{1} \frac{dy}{f(y)} \quad ?$$

   What then is T, symbolically and numerically?

(h) Now for some fun... go to the website `www.wolframalpha.com` and enter in the search box the command `integrate ZZZZZZZ dy from y=-1 to +1`, but, replace the `ZZZZZZZ` with your *integrand* $1/f(y)$. Raising y to a power of n is done with a caret, `^`, like `y^4` for y^4, and a square root is entered as `sqrt`... thus $1/\sqrt{1-y^4}$ would be entered as `1/sqrt(1-y^4)` What do you get?

(i) Now the most fun (and, in some ways, most difficult) part. Suppose the potential is not of the form $(1/2)kx^2$, but, instead, of the form Cx^4; this is known as a 'quartic' potential. Imagine that the constant C is chosen to be just right so that the potential energy when the particle is displaced by $-x_0$ or x_0 happens to be precisely $(1/2)kx_0^2$, the same as that for the quadratic potential. If the particle is displaced by $-x_0$ and then released, its motion will be repetitive (harmonic), but, not 'simple harmonic,' which is reserved for the $(1/2)kx^2$ potential. Plot on the energy graph of part 1b with a dotted line the quartic potential, using the numbers already given, and 'depict' the motion between the turning points as discussed in class. Also plot the $v(x)$ for the quartic potential in a dotted line on the graph of plot 1d.

(j) The most fun thing that is pretty easy to do: you can get the time it takes to go from $x=-x_0$ to $x=x_0$ for the quartic potential from `www.wolframalpha.com`. Do that.

(k) Repeat the last part two parts for a potential of the form Dx^{14}. How close is your answer, in percent, to $T_{\min}$?

2. Consider a particle of mass $m = 18/\pi^2 \approx 1.824\,\text{kg}$ moving in a potential energy of the form:

$$U(x) = \frac{a}{x^{12}} - \frac{b}{x^6},$$

with $a = 1$ Joules-meters^{12}, and $b = 2$ Joules-meters^6. Material starting on page 159 of RHK4 and on page 178 of KK should help you work this problem.

(a) Make a really good graph of $U(x)$ between $x = 3/4\,\text{m}$ and $x = 2\,\text{m}$.

(b) Find the expression for the force on the particle as a function of x, in terms of a, b, and x.

(c) At a certain $x \equiv x_e$, the force on the particle vanishes, so the particle could sit forever at x_e if undisturbed. This is known as an equilibrium position. Find x_e both symbolically (in terms of a and b), and numerically.

(d) Evaluate, numerically and symbolically, $U(x_e)$, in terms of a and b.

(e) Evaluate, numerically and symbolically, the second derivative of U(x) evaluated at $x = x_e$, also known as $U''(x_e)$. Is this quantity positive or negative?

(f) If the particle were at $x = x_e$ and were at rest, and then were displaced slightly from x_e and released, qualitatively what would happen? Is the equilibrium at $x = x_e$ stable or unstable and why?

(g) Suppose the particle is at x_e, and is then kicked so that its kinetic energy is $1/2$ Joule. What are the turning points, numerically, of the particle's motion?

(h) Graph the parabola $U(x_e) + (1/2)U''(x_e)(x - x_e)^2$ on the same graph you made earlier... do a good job!

(i) If the particle were to oscillate in the parabolic potential of the last part, what would its period be, symbolically (in terms of a, b, and m), and numerically.

3. Suppose the Earth were crushed down into a sphere that is much smaller than the Earth's current radius. Assume the amount of mass in the Earth is unchanged. At what radius (symbolically and numerically) will the escape velocity become equal to the speed of light? This radius is known as the 'Schwarzchild Radius', and the crushed Earth would be a... 'black hole'.